

State Equation and Stability for a Class of Continuous Petri Nets. Application to the Control of a Production System

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ABSTRACT

The paper presents an analysis of continuous Petri nets using linear algebra of matrices. It provides a link between the classical results of linear systems and Petri nets via the continuous model. A class of variable speed continuous Petri nets (VCPNs), namely Extended State Machines VCPNs (ESMVCPNs) is studied. A state-variable representation and its associated matrix algebra are used to describe systems modeled by ESMVCPNs. The state variable formulation introduced here is that of a linear continuous time-invariant system with nonnegative state (marking) and control vector. The stability and asymptotic stability concepts of linear dynamic systems are applied to continuous Petri nets. A necessary and sufficient condition of conservativeness of ESMVCPNs, based on eigenvalues, is proved. These results are used for the control of a production system.

KEY WORDS: Continuous Petri Nets, State equation, Stability, Modeling, Control, Analysis, Eigenvalues, Production Systems.

1. Introduction

The complex nature of modern industrial systems, such as production systems, has motivated designers to develop methods and tools for the modeling, analysis, performance evaluation, and control of such systems. One of the most successful modeling tool has been Petri nets.

Petri nets are graphical and mathematical modeling tools applicable to many systems belonging to the class of discrete event dynamic systems [1], [2], [3], [4] and [5]. As graphical tool, they provide a powerful communication medium between the engineer and the customer. As a mathematical tool, a Petri net model can be described by a set of linear algebraic equations reflecting the behavior of the system.

Continuous Petri nets have been defined by David and Alla [6], [7] and [3] to answer some modeling problems of continuous systems which cannot be modeled by (discrete) Petri nets, or modeling problems of discrete event systems to avoid the explosion of the number of reachable states in a (discrete) Petri net. The continuous approach of this modeling tool allows an analytical representation of these systems. Indeed, the marking of places and the instantaneous firing speeds of transitions are given by a set of differential equations. A continuous Petri net is, in fact, an approximation of the timed (discrete) Petri net. Two timed continuous Petri net models have been defined. The constant speed continuous Petri net (CCPN) [6] and the variable speed continuous Petri net (VCPN) [7]. CCPNs have been used in [8] and [9] for the modeling and management of production systems. In [10] and [11], Zerhouni and Alla have used VCPNs to analyse the dynamic behavior of manufacturing lines.

In this paper, a class of variable speed continuous Petri nets (VCPNs), namely Extended State Machines VCPNs (ESMVCPNs) is studied. A state-variable representation and its associated matrix algebra are used to describe systems modeled by ESMVCPNs. The state variable formulation introduced here is that of a linear continuous time-invariant system with nonnegative state (marking) and control vector. The stability and asymptotic stability concepts of linear dynamic systems are applied to continuous Petri nets. A necessary and sufficient condition of conservativeness of ESMVCPNs, based on eigenvalues, is proved. The stability conditions and the conservativeness property of ESMVCPNs are used for the control of a production system.

This paper is organized as follows. Discrete and continuous Petri nets are recalled in Section 2. The class of Extended State Machines Petri nets is introduced in Section 3 and several results relative to this subclass are reported in this section. In Section 4, a state-space formulation approach is used to describe systems modeled by ESMVCPNs. In Section 5, necessary and sufficient conditions for the stability and the conservativeness of ESMVCPNs are found. These results are used in Section 6 for the control of a production system. We conclude in Section 7.

2. Petri nets

The basic notations on Petri nets are introduced in this section following [2] and [3].

2.1. Discrete 100 nets

A non-marked Petri net is a 4-tuple $PN = \langle P, T, I, O \rangle$ where $P = \{p_1, p_2, \dots, p_n\}$ is a set of places represented by circles; $T = \{t_1, t_2, \dots, t_m\}$ is a set of transitions represented by bars; $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$; $I : P \times T \rightarrow \mathbb{N}$ is the input function such that $I(p_i, t_j)$ is the weight of the arc directed from place p_i to transition t_j , where $\mathbb{N} = \{0, 1, 2, \dots\}$; $O : P \times T \rightarrow \mathbb{N}$ is the output function such that $O(p_i, t_j)$ is the weight of the arc directed from transition t_j to place p_i . A place p_i is called an input (output) place of t_j if $I(p_i, t_j) \neq 0$ ($O(p_i, t_j) \neq 0$). A transition t_j is called an input (output) transition of p_i if $O(p_i, t_j) \neq 0$ ($I(p_i, t_j) \neq 0$). A pair of a place p_i and a transition t_j is called a self-loop if p_i is both an input and output place of t_j . A place p_i (transition t_j) is said to be a source place (source transition) if ${}^\circ p_i = \emptyset$ (${}^\circ t_j = \emptyset$). A place p_i (transition t_j) is said to be a sink place (sink transition) if $p_i^\circ = \emptyset$ ($t_j^\circ = \emptyset$), where ${}^\circ p_i$ and p_i° represent the sets of input and

output transitions of place p_i , respectively, ${}^{\circ}t_j$ and t_j° represents the sets of input and output places of transition t_j , respectively. A marking is a vector $M : P \rightarrow \mathbb{N}$ that assigns to each place of a Petri net a nonnegative integer number of tokens, represented by black dots. A marked Petri net with the given initial marking M_0 is denoted by (PM, M_0) . A transition $t_j \in T$ is enabled if and only if $M(p_i) \geq I(p_i, t_j)$, $\forall p_i \in P$. $M(p_i)$ indicates the number of tokens assigned by marking M to place p_i . An enabled transition may fire. When transition t_i fires, $I(p_i, t_j)$ tokens are removed from each input place p_i of t_j , and $O(p_i, t_j)$ tokens are added to each output place p_i of t_j . A new marking M is reached from M_0 when a firing sequence $S = t_1, t_2, \dots, t_k$ is applied to the net, we write $M_0 [S > M$. The marking M is given by $M(p_i) = M_0(p_i) + O(p_i, t_j) - I(p_i, t_j)$. The set of all reachable markings from the initial marking M_0 is represented by $R(M_0)$. The incidence matrix A of a Petri net is an $n \times m$ matrix of integers defined as $A = O - I$. The marking M which is reached when a firing sequence S is executed from M_0 satisfies the state equation $M = M_0 + A X$ where $X: T \rightarrow \mathbb{N}$ is the firing count vector of nonnegative integers. Its component x_j has fired during the execution of the firing sequence S .

The analysis of a Petri net model is performed in order to investigate the dynamic behavior of the model. There are three main analysis methods which allow behavioral and structural properties of Petri nets to be verified. Properties of Petri nets that are dependent on the initial marking are called behavioral properties. Boundedness and liveness are important behavioral properties of Petri nets. A Petri net is said to be k -bounded if each place in the net gets at most k tokens for any marking reachable from M_0 . It is said to be live if for every transition t_j and every marking M such that $M \in R(M_0)$, there exists a firing sequence from M containing t_j . Structural properties are independent of the initial marking of a Petri net. They are characterized in terms of the topological structure of the net. A Petri net is said to be structurally bounded if it is bounded for any finite initial marking M_0 . It is said to be structurally live if there exists a live marked Petri net (PN, M_0) . A Petri net is said to be conservative if, for any initial marking M_0 and a reachable marking $M \in R(M_0)$, there exists an $n \times 1$ vector Y of positive integers such that $Y^T M = Y^T M_0$. An integer solution Y of $A^T Y = 0$ is called a P -invariant. A P -invariant is said to be minimal if there is no other invariant Y' such that $Y'(p) \leq Y(p)$, $\forall p \in P$.

When time delays are associated with transitions, the Petri net model of a dynamic system is known as a discrete timed Petri net [12] and [13].

2.2. Continuous Petri nets

In a continuous Petri net, the marking of a place is no longer an integer but a real positive number. A firing of a transition is carried out like a continuous flow. In this paper, we consider VCPNs in which instantaneous firing speeds depend on the marking.

A continuous Petri net is a 6-tuple, $C = \langle P, T, V, I, O, m(0) \rangle$ where $P = \{p_1, p_2, \dots, p_n\}$ is a finite set of places; $T = \{t_1, t_2, \dots, t_m\}$ is a finite set of transitions; $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$; V is the firing speed function such that $\forall t_j \in T \rightarrow V(t_j) = V_j \in \mathbb{R}^+$, where V_j is the maximum speed of the transition t_j ; $I: P \times T \rightarrow \mathbb{R}^+$ is the input function such that $I(p_i, t_j)$ is the weight of the arc directed from p_i to t_j ; $O: P \times T \rightarrow \mathbb{R}^+$ is the output function such that $O(p_i, t_j)$ is the weight of the arc directed from t_j to p_i ; $m(0)$ is the initial marking such that $m(0): P \rightarrow \mathbb{R}^+$. An example of a continuous Petri net is given in Figure 1.

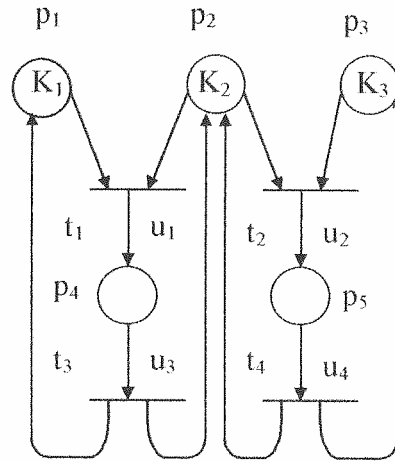


Fig. 1 A continuous Petri net

In the VCPN model, according to David and Alla [3], the maximum firing speed of a transition is, on one hand, proportional to a constant value u_j , and on the other hand, proportional to the smallest marking of its input places. If $\{p_1, p_2, \dots, p_n\}$ is the set of the input places of a transition t_i , then, the maximum firing speed $V_j(t)$ of this transition, is

$$V_j(t) = u_j \min(m_1(t), m_2(t), \dots, m_n(t)) \quad (1)$$

where u_j represents the firing frequency of t_j .

The instantaneous firing speed $v_j(t) = V_j(t)$ if and only if the transition t_j is not involved in an effective conflict. In the general case, the marking of places and the instantaneous firing speeds of transitions are given by differential equations. The variation of marking dm_i during the interval of time dt is given for each place p_i by $dm_i(t) = \sum_{j=1}^n a_{ij} v_j(t) dt$. Then for all the places of the net, we have the following fundamental equation

$$\frac{dm(t)}{dt} = Av(t) \quad (2)$$

where $m(t)=[m_1(t), m_2(t), \dots, m_n(t)]^T$ is the marking vector, $A=[a_{ij}]_{n \times m}$ is the incidence matrix, and $v(t)=[v_1(t), v_2(t), \dots, v_m(t)]^T$ is the instantaneous firing speed vector. This system of equations represents the marking behavior of every place in the net.

Consider a VCPN with m transitions and n places and assume that every transition in the net has exactly one input place. The equation (1) which represents the expression of the maximum firing speed of every transition, is reduced to

$$V_j(t) = u_j m_j(t) \quad (3)$$

where $j = 1, 2, \dots, m$ and $i = 1, 2, \dots, n$

If there is no conflict between t_j and another transition having p_i as input place, then the instantaneous firing speed $y_j(t) = V_j(t)$.

In continuous Petri nets, the definition of a P-invariant remains the same as for discrete Petri nets. That is, a vector Y is a P-invariant if it verifies the relation $Y^T A = 0$. If we multiply the fundamental equation (2) at left by Y^T , we find

$$Y^T \frac{dm(t)}{dt} = Y^T Av(t) = 0, \forall t$$

Then, by integrating the two members of this relation, we obtain $Y^T m(t) = Y^T m(0)$ where $m(t)$ is a continuous function. For example, the continuous Petri of Figure 1 has three Pinvariants $Y_1 = [10010]^T$, $Y_2 = [00101]^T$ and $Y_3 = [0101]^T$. That gives three linear invariants of markings, that is $m_1(t) + m_4(t) = K_1$, $m_3(t) + m_5(t) = K_3$, $m_2(t) + m_4(t) + m_5(t) = K_2$.

3. Extended State Machines Petri nets

3.1. Definitions

Finite-state machines or their diagrams can be equivalently represented by a class of Petri nets called state machines Petri nets. This subclass has been widely studied and its properties are well known. For instance, Murata [2] reports several results on the properties of this class of nets.

DEFINITION 1: A state machine is a Petri net such that each transition has exactly one input arc and one output arc. This can be expressed concisely as

$$|\overset{\circ}{t}_j| = |t_j^{\circ}| = 1 \quad \forall t_j \in T$$

State machines may represent choice, since a place may have more than one outgoing arc, but cannot model synchronization or concurrency.

The model considered in this paper is based on state machine Petri nets. To describe concurrent systems, the model is extended by allowing multiple output places. The final model, called Extended State Machines (ESM) Petri nets, can model both choice and concurrent behavior, but not the synchronization of parallel activities. The ESM concept is summarized in the following definition.

DEFINITION 2: An Extended State Machine (ESM) is a Petri net such that every transition has exactly one input place.

More formally, we can say that an ESM is a Petri net such that

$$|\overset{\circ}{t}_j| = 1 \quad \forall t_j \in T$$

The class of ESM Petri nets was first defined by Holt and Commoner [14], No results related to it were reported there. In [15], Lien has defined this subclass and further studied the concept of strong-connectedness, consistency, and conservativeness. Some properties of this subclass were also studied by Memmi in [16] and [17]. Recently, this subclass has been studied in [18], [19], and [20] and a complete structure theory, based on eigenvalues, has been developed.

The results presented in [18], [19], and [20] on structural boundedness, conservativeness, consistency, liveness, repetitiveness, and well-structuredness of this class of nets follow from the computation of eigenvalues of a square matrix, known as an M-matrix, obtained by a transformation of the classical incidence matrix. This transformation results on a square matrix with nonnegative off-diagonal elements. The modified incidence matrix eigenvalues are computed and Theorems on matrices of this type are used to prove sufficient and/or necessary conditions for some structural properties of ESM Petri nets.

Consider, for instance, the ESM Petri net of Figure 2.

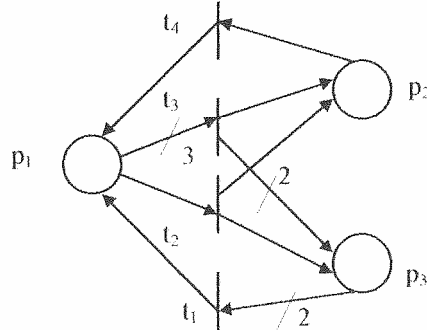


Fig. 2. An ESM Petri net

The modified incidence matrix D associated with this net is:

$$D = \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{array}{ccc} \left[\begin{array}{ccc} -4 & 1 & 1 \\ 2 & -1 & 0 \\ 3 & 0 & -2 \end{array} \right] \end{array} \begin{array}{l} p_1 \\ p_2 \\ p_3 \end{array} \end{array}$$

In this matrix, a row and a column are both associated with places. Each row corresponds to the marking modification of a place when its input and output transitions are fired. For example, the first row means that p_1 loses four tokens when its output transitions are fired and receives one token when the output transition of p_2 (respectively p_3) is fired.

We deduce the following relation

$$D = AU^T \quad (4)$$

where D is the $n \times n$ modified incidence matrix of the ESM Petri net of Figure 2, A is the $n \times m$ incidence matrix and $U = [u_{ij}]_{n \times m}$ is defined as follows [20].

$$u_{ij} = \begin{cases} 1 & \text{if } p_i \in {}^\circ t_j \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Since every transition of an ESM Petri net has exactly one incoming arc (i.e. one and only one input place), then each column of the matrix U contains one and only one positive element, for instance unity.

3.2. Structural boundedness crud conservativeness

First we shall adopt some basic definitions in linear systems theory. Let denote by $\det(D)$ the determinant of an $n \times n$ matrix D . It is well known [21] that a scalar λ is an eigenvalue of an $n \times n$ matrix D if and only if it is a solution of $\Delta(\lambda) = \det(\lambda I - D) = 0$, where I is the unit matrix of order n . $\Delta(\lambda)$ is a polynomial of degree n in λ and is called the characteristic polynomial of D . Since $\Delta(\lambda)$ is of degree n , the $n \times n$ matrix D has n eigenvalues (not necessarily all distinct). Any nonzero vector X satisfying $DX = \lambda X$ is called an eigenvector of D associated with the eigenvalue λ .

Let D be the modified incidence matrix of an ESM Petri net. Let denote by Z the class of all real square matrices whose off-diagonal elements are all nonpositive. Then $-D \in Z$. The following Theorems due to Fiedler and Ptak [22], are used in this paper to prove the structural boundedness, structural liveness, and conservativeness of ESM Petri nets.

THEOREM 1: Let $D \in Z$ and suppose that there exists a vector $X > 0$ such that $DX \geq 0$, then all real eigenvalues of D are nonnegative.

THEOREM 2: Let $D \in Z$, then the following conditions are equivalent to each other:

1. there exists a vector $X \geq 0$ such that $DX > 0$
2. there exists a vector $X > 0$ such that $DX \geq 0$
3. the real part of each eigenvalue of D is positive
4. all real eigenvalues of D are positive

Propositions 3, 4, and 6, and corollary 5 are necessary conditions for structural boundedness, conservativeness, and structural liveness of ESM Petri nets [18], [19], and [20].

PROPOSITION 3: If an ESM Petri net, N , is structurally bounded, then all real eigenvalues of D are nonpositive.

Proof : Suppose N is structurally bounded. Then there exists an n -vector Y of positive integers such that $A^T Y \leq 0$. Multiplying both sides by U gives $UA^T Y \leq 0$, or $D^T Y \leq 0$, or $-D^T Y \geq 0$. The conditions of Theorem I may be applied to $-D^T$. Indeed, if there exists a vector $Y > 0$ such that $-D^T Y \geq 0$ then all real eigenvalues of $-D$ are nonnegative (or, equivalently, all real eigenvalues of D are nonpositive).

PROPOSITION 4 : If an ESM Petri net, N , is conservative, then zero is a simple eigenvalue of D and all other real eigenvalues of D are negative. The minimal P -invariant Y is the n -eigenvector of D^T of positive integers associated with $\lambda = 0$.

Proof: Suppose N is conservative. Then there exists an n -vector Y of positive integers such that $A^T Y = 0$. Multiplying both sides by U gives $UA^T Y = 0$, or $D^T Y = 0$. According to Theorem 1, all real eigenvalues of $-D$ are nonnegative (or equivalently all real eigenvalues of D are nonpositive). Let the n -vector Y of positive integers be the eigenvector of D^T associated with the eigenvalue λ such that $D^T Y = \lambda Y = 0$. Hence, zero is a simple eigenvalue of D .

COROLLARY 5 : If D is nonsingular then N is not conservative.

Proof : Since $\det D \neq 0$, no eigenvalue of D can be zero. Obviously if all the eigenvalues of D are different from zero, the rank of D is equal to n . Then the only solution of $D^T Y = 0$ is $Y = 0$. Hence N is not conservative.

PROPOSITION 6: Let N be a structurally bounded ESM Petri net. If all real eigenvalues of D are negative, then (N, M_0) is not live.

Proof: If there exists a vector X of nonnegative integers such that $AX \leq 0$, then a Petri net is not live. According to Theorem 2, if all real eigenvalues of $-D$ are positive (or equivalently all real eigenvalues of D are negative), then there exists a vector $x \geq 0$ such that $-Dx > 0$. Let $D = AU^T$, then $-AU^T x > 0$ or $AU^T x < 0$. Let $X = U^T x \geq 0$. Then there exists $X \geq 0$ such that $AX < 0$. These results have been proven in [20]. The proofs are included here just for the completeness of this paper.

4. State space formulation approach

The definition of ESM Petri nets can be extended to VCPNs in which each transition has exactly one input place- Like ESM Petri nets, this class, called Extended state machines VCPNs (ESMVCPNs for short), is a class of variable speed continuous Petri nets that cannot model synchronization.

In this section, a state space formulation approach is used to describe systems modeled by ESMVCPNs. The state equation representation is based on the following assumption.

ASSUMPTION 1: Each transition fires at its maximal speed $V_j(t)$.

Let p_i in Figure 3 represents the i th place of an ESMVCPN of n places; m_i is the marking of the place p_i . Let $i_i(t)$ be the constant firing speed of the transition t_i , where the pair (p_i, t_i) is a self-loop. Let u_{0i} , u_{ji} , and u_{ij} be respectively the constant firing speeds of the sink transition t_{0i} , the transition t_{ji} , and the transition t_{ij} . According to Assumption I no transition in the Petri net of Figure 2 is involved in an effective conflict, then the instantaneous firing speeds of t_{0i} , t_{ji} , and t_{ij} , where m_j represents the marking of the place p_j , are given by

$$\begin{aligned} V_{0i} &= u_{0i} m_i \\ V_{ji} &= u_{ji} m_i \\ V_{ij} &= u_{ij} m_j \end{aligned} \quad (6)$$

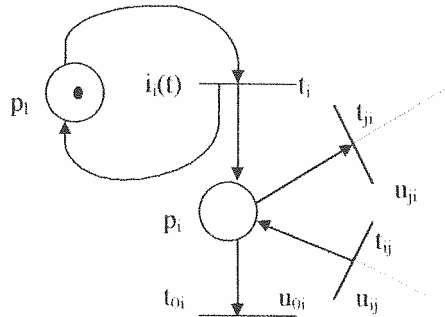


Fig. 3. The i th place of an ESMVCPN of n places

The equation for the i th place is therefore

$$\frac{dm_i(t)}{dt} = -u_{oi}m_i(t) - \sum_{j \neq i} u_{ij}m_i(t) + \sum_{j \neq i} u_{ji}m_j(t) + i_i(t) \quad (7)$$

Let

$$u_{ii} = \left[u_{oi} + \sum_{j \neq i} u_{ji} \right] \quad (8)$$

then

$$\frac{dm_i(t)}{dt} = -u_{ii}m_i(t) + \sum_{j \neq i} u_{ij}m_j(t) + i_i(t) \quad (9)$$

where $i = 1, 2, \dots, n$

Thus, in matrix form

$$\dot{m}(t) = u.m(t) + i(t) \quad (10)$$

where,

$$u = \begin{bmatrix} -u_{11} & u_{12} & \cdot & u_{1m} \\ u_{21} & -u_{22} & \cdot & u_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \cdot & -u_{nm} \end{bmatrix}; \quad m(t) = \begin{bmatrix} m_1(t) \\ m_2(t) \\ \cdot \\ m_n(t) \end{bmatrix}; \quad i(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \\ \cdot \\ i_n(t) \end{bmatrix} \quad (11)$$

For a system without input, $i(t) = 0$, the non-homogeneous equation (10) becomes (12), where u is the $n \times n$ evolution matrix shown in (11).

$$\dot{m}(t) = u.m(t) \quad (12)$$

u can be computed as follows

$$u = AU^T \quad (13)$$

where A is the incidence matrix of the net and $U = [u_{ij}]_{n \times m}$ such that

$$u_{ij} = \begin{cases} u_j & \text{if } p_i \in {}^\circ t_j \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

where u_j is the firing frequency of transition t_j .

REMARK: The number of self-loop transitions and sink transitions that may be associated to each place of an ESMVCPN is not limited, and any transition involved in a self-loop may have multiple output places. This can be shown on the following example.

Consider place p_i of the ESMVCPN given in Figure 4.

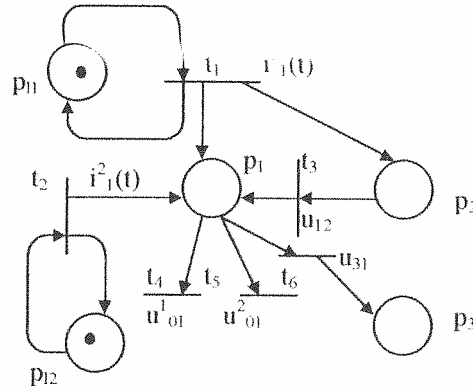


Fig.4 An ESMVCPN with two self-loop transitions and two sink transitions

The equation for the place p_1 is given by

$$\frac{dm_1(t)}{dt} = -u^1_{01}m_1(t) - u^2_{01}m_1(t) + u_{31}m_1(t) + u_{12}m_2(t) + i^1_1(t) + i^2_1(t)$$

$$\text{or } \frac{dm_1(t)}{dt} = -u_{01}m_1(t) - u_{31}m_1(t) + u_{12}m_2(t) + i_1(t)$$

where $u_{01} = u_{01}^1 + u_{02}^2$ and $i_1(t) = i_1^1(t) + i_1^2(t)$

Let $u_{11} = u_{01} + u_{31}$

$$\text{then } \frac{dm_1(t)}{dt} = -u_{11}m_1(t) + u_{12}m_2(t) + i_1(t)$$

which corresponds to equation 9.

5. Stability

When a physical system is modeled, the next step is to develop mathematical equations that describe it. Once this mathematical description is obtained, a quantitative and/or a qualitative analysis of the system is done. In the qualitative analysis, we are interested by the general properties of the system such as the stability that we introduce in this section. The stability concept of dynamic systems is applied to continuous Petri nets.

The following stability definition of an ESMVCPN is equivalent to the boundedness definition of a continuous Petri net such as presented in [6].

DEFINITION 3 : An ESMVCPN is said to be stable if there exists a finite constant $C > 0$ such that, for any $t \geq 0$, the marking of the place p_i , $m_i(t) \leq C$ for $i=1, 2, \dots, n$.

DEFINITION 4 : An ESMVCPN is said to be asymptotically stable if $\lim_{t \rightarrow \infty} m_i(t) = 0$, $i=1, 2, \dots, n$.

THEOREM 7 : An ESMVCPN is stable iff the eigenvalues of its evolution matrix u have nonpositive (negative or zero) real parts. The eigenvalue with a zero real part must be unique. In the case where all the eigenvalues of u have negative real parts, the ESMVCPN is asymptotically stable.

Proof: The solutions of the state equation (12) are of the form $m(t) = m(0)e^{ut}$, where $m(0)$ is the initial marking vector of the ESMVCPN. In order to have $m_i(t) \leq C$, with $C > 0$, and $i = 1, 2, \dots, n$, e^{ut} must be bounded for all $t \geq 0$. Let P be the nonsingular matrix such that $\hat{u} = PuP^{-1}$ and \hat{u} is in the Jordan form. Since $e = Pe^{ut}P^{-1}$, then $\|e^{\hat{u}t}\| \leq \|P\| \|e^{ut}\| \|P^{-1}\|$. Consequently, if $\|e^{ut}\|$ is bounded, so is $\|e^{\hat{u}t}\|$. Conversely, from the equation $e^{ut} = Pe^{\hat{u}t}P^{-1}$, we see that if $\|e^{\hat{u}t}\|$ is bounded, so is $\|e^{ut}\|$. Hence, $m_i(t) \leq C$ iff $\|e^{\hat{u}t}\|$ is bounded. $\|e^{\hat{u}t}\|$ is bounded if every entry of $e^{\hat{u}t}$ is bounded. Every entry of $e^{\hat{u}t}$ is of the form $t^{k_j-1} e^{c_j t + i\omega_j t}$

where $\lambda_j = \alpha_j + i\omega_j$ is an eigenvalue of u of multiplicity k .

It is easy to see that $\|e^{\hat{u}t}\|$ is bounded when $\alpha_j < 0$. In the case where $\alpha_j = 0$, $\|e^{\hat{u}t}\|$ is bounded iff $k_j = 1$. If all the eigenvalues of A have negative real parts, we conclude that $\lim_{t \rightarrow \infty} e^{\hat{u}t} = 0$, consequently $\lim_{t \rightarrow \infty} m(t) = 0$. The ESMVCPN is, in this case, asymptotically stable.

To show the uniqueness of the zero real part, consider the following example.

Example 1: Figure 5 shows an unstable ESMVCPN.

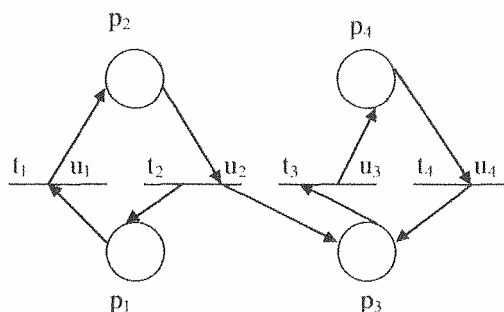


Fig. 5 An unstable ESMVCPN

The corresponding evolution matrix u is given by

$$u = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ -u_1 & u_2 & 0 & 0 \\ u_1 & -u_2 & 0 & 0 \\ 0 & u_2 & -u_3 & u_4 \\ 0 & 0 & u_3 & -u_4 \end{bmatrix} \begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{matrix}$$

The eigenvalues of u are the roots of $\Delta(\lambda) = \det(\lambda I - u) = 0$, where I is a 4×4 identity matrix.

If $u_1 = u_2 = u_3 = u_4 = 1$, we have $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -2, \lambda_4 = -2$.

The eigenvalue with a zero real part is not unique, the ESMVCPN is thus unstable.

The property of conservative of (discrete) Petri net can be generalized without difficulties to continuous Petri net.

DEFINITION 5 : An ESMVCPN is said to be conservative if there exists a positive real number $y_i(p)$ for every place p_i such that $\sum_{i=1}^n y_i m_i(t) = \text{a constant}$ for any $t \geq 0$ and for any initial marking $m(0)$ of the ESMVCPN.

THEOREM 8 : An ESMVCPN is conservative iff there exists an eigenvector Y of u^T of positive real numbers associated with a zero eigenvalue λ of the evolution matrix u such that $u^T Y = \lambda Y = 0$.

Proof: Suppose there exists an eigenvector $Y > 0$ of u^T associated with $\lambda = 0$ such that $u^T Y = Y^T u = \lambda Y = 0$, with $Y = [y_1, y_2, \dots, y_n]^T$. Multiplying both sides of this relation at right by $m(t)$,

with $m(t) = [m_1(t) \ m_2(t) \ \dots \ m_n(t)]^T$, gives $Y^T u m(t) = 0$. Since $\dot{m} = u m(t)$, then $Y^T \dot{m}(t) = 0$, hence $Y^T m(t) = a$ constant. Conversely, suppose the ESMVCPN is conservative. Then there exists $Y > 0$ such that $A^T Y = 0$. Multiplying both sides at left by U gives $U A^T Y = 0$. Since $u = A U^T$, then $u^T Y = \lambda Y = 0$. Example 2 shows that $\lambda = 0$ must be unique.

There is a similarity between the stability of dynamic systems (the euclidean norm is bounded) and the boundedness of Petri nets (the marking vector is required to be bounded). The condition of stability (boundedness) is necessary and sufficient in the case of continuous Petri nets. It is a necessary condition for discrete Petri nets. This can be shown by the following example.

Example 2: Let the Petri net shown in Figure 6 be an ESMVCPN.

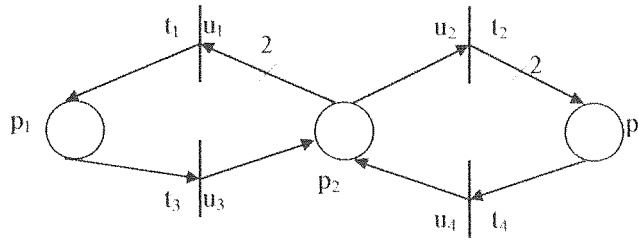


Fig.6 A stable ESMVCPN

For $u_1 = u_2 = u_3 = u_4 = 1$, the evolution matrix is given by

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 0$, and $\lambda_3 = -4$.

According to Theorem 7, the ESMVCPN is stable. Figure 7 shows the evolution of the markings $m_1(t)$, $m_2(t)$, and $m_3(t)$.

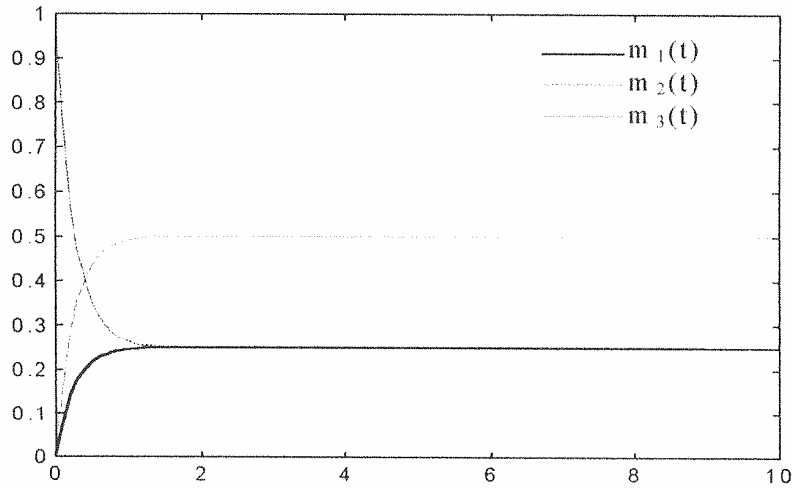


Fig.7 Evolution of the markings of the ESMVCPN of Fig. 6

This is not the case if we suppose that the Petri net of Figure 6 is a discrete Petri net. Place p_3 is indeed unbounded. The evolution of the markings is given in Figure 8 where the marking of place p_3 increases.

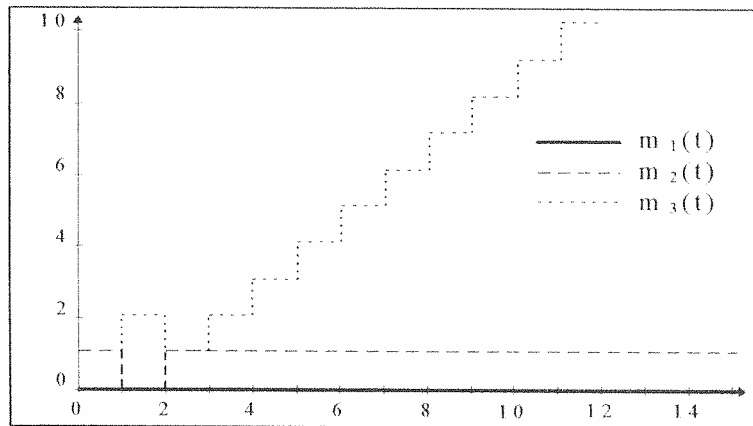


Fig. 8 Evolution of the markings in the discrete case

This difference between the continuous model and the discrete model may be explained by the fact that in the continuous case, the token in place p_2 is supposed to be constituted of very small parts (continuous parts). Consequently, the set $\langle p_2, \{t_1, t_2\} \rangle$ is not an effective conflict. As long as the marking m_2 is different from zero, transitions t_1 and t_2 are both enabled, and fire at their maximal speed.

6. A production system

Consider the Petri net model (see Figure 10) of a production system shown in Figure 9. The production system in this example consists of two workstations, each capable of producing two part-types, called A and B. The part-types are stored in two different output buffer stocks. The two workstations need common resources to carry out their operations. The resources consist of raw materials, human resources, and material resources. The human resources expenses correspond to the staff wages, and the material expenses correspond to the investment and maintenance. The raw materials expenses correspond to a regular supplying with a given price and a maximum speed. All these expenses are made possible by common funds regularly feeded by the results of the sale of the produced parts.

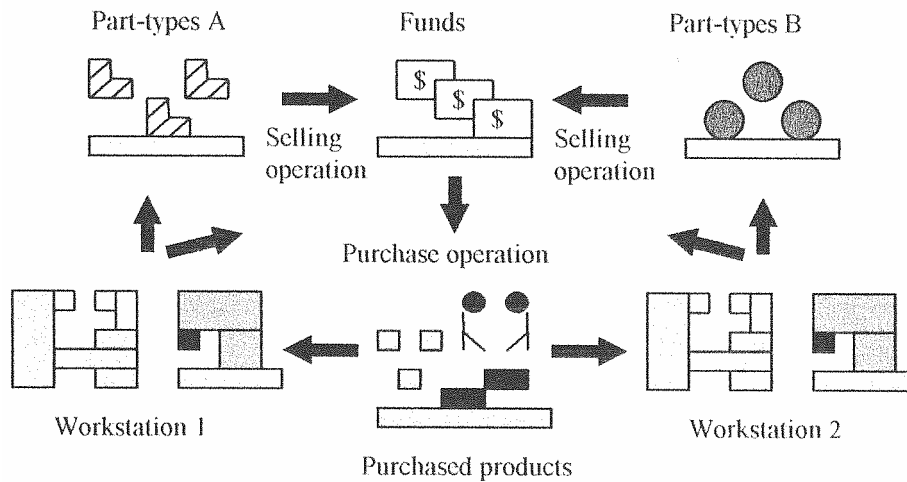


Fig.9 A production system consisting of two workstations

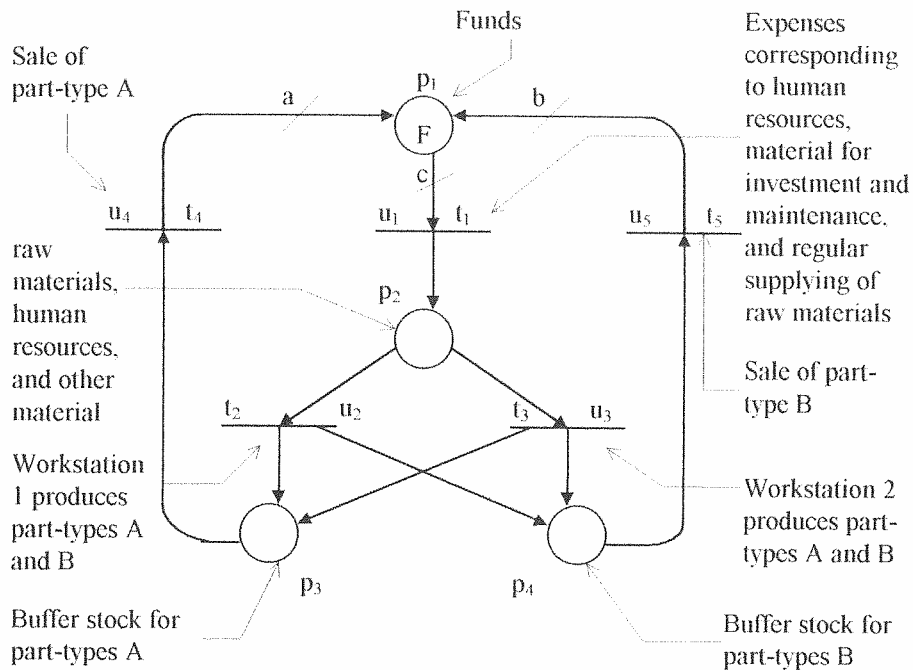


Fig. 10 A continuous Petri net model of a production system

In the initial state, the only available entity are the funds modeled by place p_1 such that $m_j(0)=F$. $V_1(t)=u_1m_1(t)$ is the maximum firing speed associated to transition t_1 . It corresponds to the maximum amount of money (raw materials, human resources, and other material expenses) we can spend in one time unit. Workstations 1 and 2 are modeled by transitions t_2 and t_3 respectively. The associated maximum firing speeds $V_2(t)=u_2m_2(t)$ and $V_3(t)=u_3m_2(t)$ denote the amounts of produced parts in one time unit. Transitions t_2 and t_3 represent the processing operations while transitions t_4 and t_5 correspond to the selling operations of the produced parts such that $V_4(t)=u_4m_3(t)$ (respectively $V_5(t)=u_5m_4(t)$) is the maximum amount of part-types A (respectively B) sold in one time unit. The produced parts are stored in output buffer stocks modeled by places p_3 , and p_4 .

Weight a (respectively b) corresponds to the unit price of part-types A (respectively B). Weight c represents the unit price of purchased products.

The differential equations describing the system are given below.

$$\dot{m}_1(t) = -cu_1m_1(t) + au_4m_3(t) + bu_5m_4(t)$$

$$\dot{m}_2(t) = u_1m_1(t) - u_2m_2(t) - u_3m_2(t)$$

$$\dot{m}_3(t) = u_2 m_2(t) + u_3 m_2(t) - u_4 m_3(t)$$

$$\dot{m}_{41}(t) = u_3 m_2(t) + u_2 m_2(t) - u_5 m_4(t)$$

or in matrix form

$$\begin{bmatrix} \dot{m}_1(t) \\ \dot{m}_2(t) \\ \dot{m}_3(t) \\ \dot{m}_4(t) \end{bmatrix} = \begin{bmatrix} -cu_1 & 0 & au_4 & bu_5 \\ u_1 & -(u_2 + u_3) & 0 & 0 \\ 0 & (u_2 + u_3) & -u_4 & 0 \\ 0 & (u_2 + u_3) & 0 & -u_5 \end{bmatrix} \begin{bmatrix} m_1(t) \\ m_2(t) \\ m_3(t) \\ m_4(t) \end{bmatrix}$$

which may be written as

$$\dot{m}_1(t) = um(t)$$

where u is the evolution matrix of the ESMVCPN or Figure 10 and $m(t)$ the marking vector. The eigenvalues are the roots of the following characteristic equation

$$\Delta(\lambda) = \det(\lambda I - u) = (\lambda + cu_1) [\lambda + (u_2 + u_3)] [\lambda + u_4] (\lambda + u_5) - au_4 [u_1(\lambda + u_5)(u_2 + u_3)] - bu_5 [u_1(\lambda + u_5)(u_2 + u_3)] = 0$$

According to Theorem 8, an ESMVCPN is conservative iff $\lambda=0$. Consider the above characteristic equation and let $\lambda=0$. We have

$$Sf u_1 u_4 u_5 (u_2 + u_3) [c - (a + b)] = 0$$

Since $u_j \geq 0$, then $c = (a + b)$ This is the limit case.

Let, for example, $u_1 = u_2 = u_3 = u_4 = u_5 = 1$, and $c = 3$.

case 1: $c = a + b$

Let the unit price of part-types A be $a = 2$, and the unit price of part-types B be $b = 1$. The evolution matrix is

$$u = \begin{bmatrix} -3 & 0 & 2 & 1 \\ 1 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

The eigenvalues, solutions of the characteristic equation, are

$$\lambda_1 = -3 + 1.41i, \lambda_2 = -3 - 1.41i, \lambda_3 = 0, \text{ and } \lambda_4 = -1$$

The ESNAVCPNf is conservative. The P-invariant is the eigenvector Y of u^T associated with $\lambda_3 = 0$. That is $Y = [1 \ 3 \ 2 \ 1]^T$. We can express this as $m_1(t) + 3m_2(t) + 2m_3(t) + m_4(t) = F$. *This is the limit case.*

The markings evolution is represented in Figure 11,

case 2: $c > a + b$

Let $a = b = 1$.

The eigenvalues are $\lambda_1 = -2.89 + 1.19i$, $\lambda_2 = -2.89 - 1.19i$, $\lambda_3 = -0.20$, and $\lambda_4 = -1$.

The ESMVCPN is asymptotically stable and, according to Proposition 6, it is not live. Consequently $\lim_{t \rightarrow \infty} m(t) = 0$. *The firm makes no profits.* Figure 12 shows the evolution of the markings for the case $c > a + b$.

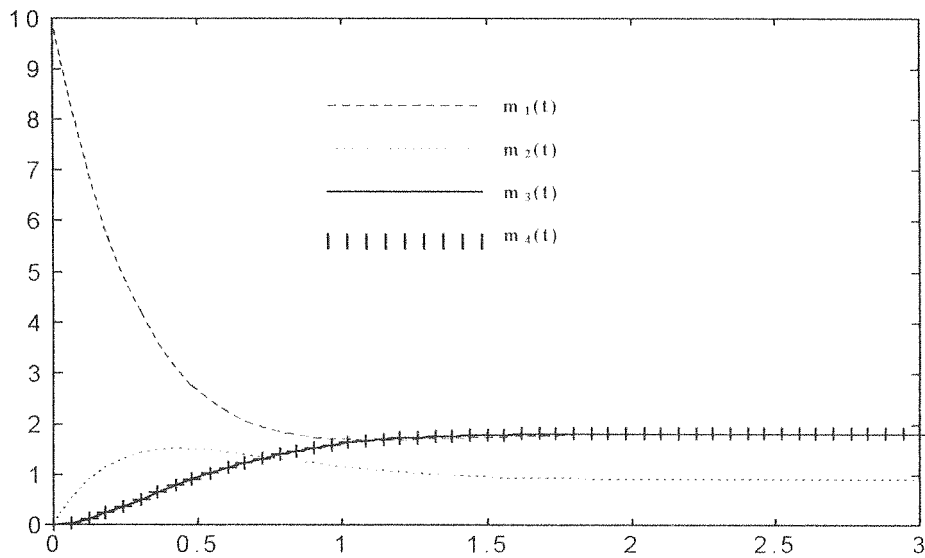


Fig. 11 The markings evolution for the case $c = a + b$

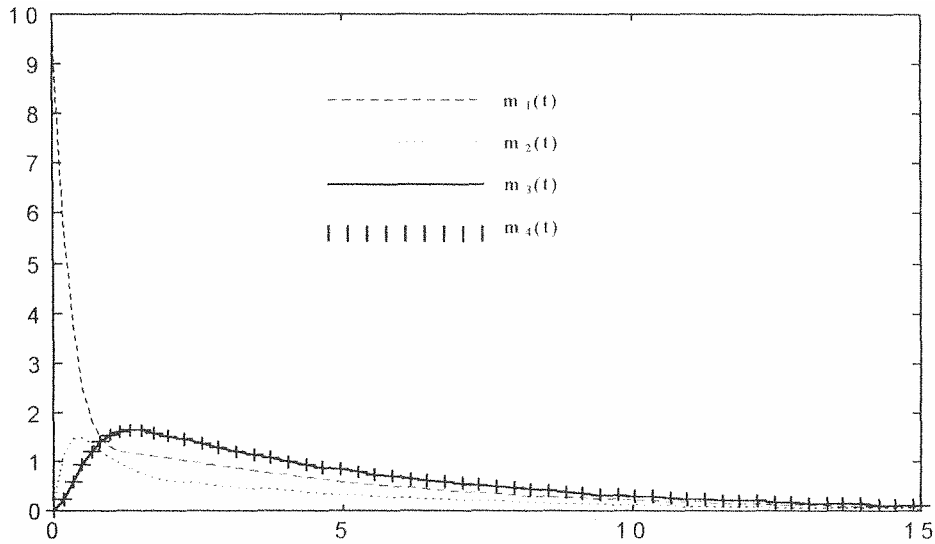


fig. 12 The marking evolution for the case $c > a + b$

case 3: $c < a + b$

Let $a = b = 2$.

The eigenvalues are $\lambda_1 = -3.08 + 1.58i$, $\lambda_2 = -3.08 - 1.58i$, $\lambda_3 = +0.16$, and $\lambda_4 = -1$. One of the eigenvalues is real and positive. The ESMVCPN is not stable, hence $\lim_{t \rightarrow \infty} m(t) \rightarrow \infty$.

As long as $c < a + b$, the firm makes profit. Increasing a , the unit price of part-types A, and/or b , the unit price of part-types B, or decreasing c , the unit price of purchased products, increases profits. The markings evolution for the case $c < a + b$ is shown in Figure 13.

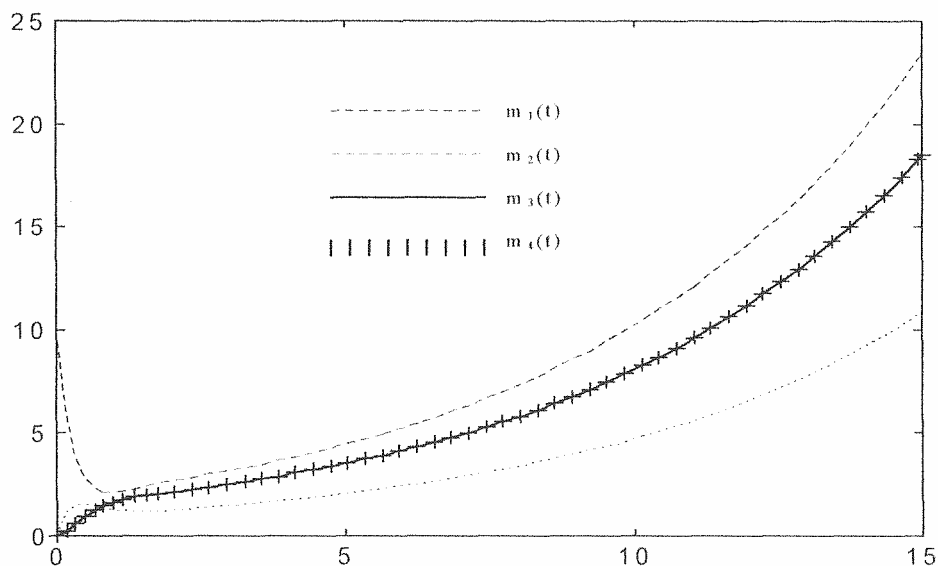


Fig. 13 The markings evolution for the case $c < a+b$

7. Conclusion

An analysis of a class of continuous Petri nets using linear algebra of matrices has been presented in this paper. It provides a link between the classical results of linear systems and Petri nets via the continuous model. The concepts of stability, asymptotic stability, conservativeness of ESMVCPNs have been used for the control of a production system. It is an interesting application example of continuous Petri nets because entities such as funds, resources, and produced parts may occur in large quantities. Consequently, the explosion problem of the number of reachable states in a (discrete) Petri net is avoided. The control method presented in this paper is potentially powerful for relatively small systems. More research is needed to evaluate the method's performance for large manufacturing systems.

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