# Bootstrap-Based Test for Stationarity and Cointegration

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**Abstract**

This paper investigates the small sample properties of a bootstrap-based test for the null of trend stationarity and cointegration. We consider the KPSS test for stationarity by Kwiatkowski, Phillips, Schmidt, and Shin [16] and the residual-based test for cointegration of Shin [32]. Our simulation results show that the asymptotic approximations of the limiting null distributions do not perform satisfactory in small to moderate samples. As an alternative, we consider a bootstrap approach. Validity of the bootstrapped tests is studied and a Monte Carlo experiment is conducted to evaluate the applicability of the proposed bootstrap procedures.

## 1. Introduction

There is a large literature in time series econometrics on the debate about whether economic time series are best characterized as trend stationary processes or difference stationary processes. Since the influential article [23], hundreds of economic time series have been examined by unit root tests. Empirical evidence has accumulated that many economic and financial time series contain a unit root ([1], [12], [22], and [26], among others). However, as argued elsewhere (see for example Kwiatkowski et al. [16]), most standard testing procedures consider the null hypothesis of a unit root which ensures that the null hypothesis is accepted unless there is strong evidence against it. Monte Carlo evidence ([5], [7], [30], and [34]) show that the discriminatory power of unit root tests is often low, indicating standard unit root tests are not very powerful against trend stationary alternatives. Indeed, different results have been obtained from other approaches.

Given these empirical results and Monte Carlo evidence, to decide whether a time series is trend stationary or difference stationary, it would be useful to perform tests for the null hypothesis of stationarity as well as tests for a unit root. However, although the literature on testing the null hypothesis of a unit root is huge (see, inter alia, [6] and [28]), there have been only several attempts on testing stationarity ([16], [19], [25], and [35]). In particular, Kwiatkowski et al. [16] (hereafter KPSS) considered a time series model that can be decomposed as the sum of a deterministic trend, a random walk, and a stationary error, and proposed an LM test for the null hypothesis of stationarity. A similar test which differs from the KPSS test in its treatment of autocorrelation and applies when the null hypothesis is an AR($k$) process is suggested in [19]. A stationarity test by examining the fluctuations in the detrended time series is proposed in [35]. A Kolmogoroff-Smirnoff type test for stationarity was given and compared with the KPSS test. In particular, notice that the KPSS test has the Cramer-von Mises limiting distribution and can be represented as an infinite weighted sum of independent central chi-squared random variables, both the test of [35] and the KPSS test [16] can be obtained by testing the fluctuations in the detrended time series.

The related issues in multivariate time series have also attracted a good deal of research, and the univariate unit root tests and stationarity tests have been extensively used to test for the presence of cointegration using residual based approaches. The tests are used in the same way as standard unit root tests and stationarity tests, but the data are the residuals from a least squares cointegrating regression (OLS regression among the levels of economic time series) ([9] and [27]). Among various tests for cointegration, the residual-based procedure has been one of the most frequently used approaches in empirical research.

The majority of the residual based cointegration tests is designed to test the null of no cointegration. However, testing the null of cointegration is more intuitive than testing the null of no cointegration, because more often we are interested in the cointegration relationship. There are relatively fewer approaches that advocate testing the null of cointegration ([25], [32], and [36]). One residual based approach that is frequently used in testing the null of cointegration was proposed in [32] where the cointegration regression model and the test statistics for the null of cointegration are discussed. The asymptotic distribution of the test statistics and simulated critical values are reported as well. It should be noted that although being simple and straightforward, in small to moderate samples, the performance of this test procedure is unknown. Given the fact that many asymptotic test procedures do not perform well in small to moderate samples, it is therefore important to investigate this issue.

We believe that the bootstrap method can address some of the above issues. Although the use of resampling in statistical inference is not new (see [37]), the expansion of this idea to a wide variety of statistics starts since [8]. Bootstrap was extended to regression models in [10] and [11]. The literature in the application of the bootstrap method is huge, including [13], [14], and [15]. Bootstrap methods have been used in the analysis of nonstationary time series models. Bootstrap methods were used to deal with the estimation and inference of cointegrating regressions in [20]. The asymptotic property of the bootstrap procedures for cointegration regressions was discussed in [21] where some theoretical results regarding the asymptotic validity of the proposed bootstrap procedures are provided.

The purpose of the paper is twofold. First, we perform a simulation study to examine the small sample behavior of the KPSS test for stationarity and the Shin test for cointegration. The simulation results show that the asymptotic approximation does not work satisfactory in small samples. Second, we consider the bootstrap test as an alternative. The bootstrap procedure and its asymptotic properties are studied. Since the critical values in the bootstrap test is sample or data dependent, we expect that there will be a certain degree of small sample performance improvement using the bootstrap approach. Monte Carlo results are provided to evaluate the performance of the proposed procedure.

The rest of the paper is organized as follows. Section 2 discusses the stationarity test and Section 3 proposes a bootstrap-based test for this hypothesis. Cointegration test is studied in Section 4. The bootstrap procedure is discussed and the asymptotic validity of the proposed bootstrap is developed as well. In section 5 we provide some Monte Carlo simulation evaluating the KPSS and Shin test and the proposed bootstrap procedure. Finally, section 6 concludes.

## 2. Testing Stationarity

One important reason for the large literature on estimation and hypothesis testing in time series regressions with integrated processes is that many observed time series display nonstationary characteristics. With the majority of the unit root tests considering the null of a unit root, several studies investigate the issue of testing the null of stationarity, e.g., [2], [3], [4], [16], [24], and [35].

Kwiatkowski *et al.* (1992) develop a unit root test with the null hypothesis being that a time series is stationarity around a deterministic trend. They choose a components representation of the time series $y_t$ ($t = 1, \ldots, T$) which is decomposed as the sum of deterministic trend, random walk, and stationary error. The stochastic trend $y_t^s$ ($y_t^s = y_{t-1}^s + u_t$) is annihilated when $\sigma_u^2 = \mathrm{var}(u_t) = 0$, which therefore corresponds to a null hypothesis of trend stationarity. Under Gaussian assumptions and iid error conditions, the hypothesis can be tested in a simple way using the *LM* principle. This procedure can easily be extended to more general cases where there is serial dependence by replacing the estimate of the variance parameter of the stationary error with corresponding estimates of the long run variance. This was done in KPSS, where a general approach was developed.

Consider a time series $y_t$ which is decomposed as the sum of deterministic trend $h_t$, random walk $y_t^s$, and stationary component $v_t$

$$y_t = h_t + y_t^s + v_t \,, \; h_t = \gamma' x_t \,, \; y_t^s = y_{t-1}^s + u_t \,. \tag{1}$$

The deterministic trend $h_t$ depends on unknown parameters and is specified as $h_t = \gamma' x_t$ where $\gamma = (\gamma_0, ...., \gamma_p)'$ is a vector of trend coefficient and $x_t$ is a deterministic trend of known form, say, $x_t = (1, t, ..., t^p)'$. The leading cases of the deterministic component are (i) a constant term $x_t = 1$; and (ii) a linear time trend $x_t = (1, t)'$. We assume that there is a standardizing matrix $D_T$ such that $D_T x_{[Tr]} \to X(r)$ as $n \to \infty$. For the case of a linear trend, $D_T = diag[1, T^{-1}]$ and $X(r) = (1, r)'$. More generally, if $X_t$ is a polynomial trend that $x_t = (1, t, ..., t^p)'$, $D_T = diag[1, T^{-1}, ...., T^{-p}]$, and $X(r) = (1, r, ..., r^p)'$.

The stochastic trend $y_t^s$ is annihilated when $\sigma_u^2 = \mathrm{var}(u_t) = 0$, corresponding to the null hypothesis of (trend) stationarity. The stationary component $v_t$ is an $AR(q)$ process that

$$\Psi(L) v_t = w_t \,, \tag{2}$$

where $\{w_t\}$ are iid random variates with mean zero and variance $\sigma_w^2$, $\Psi(L)$ is a $q$-th order polynomial of the lag operator $L$ defined as

$$\Psi(L) = 1 - \Gamma_1 L - \Lambda - \Gamma_q L^q \,. \tag{3}$$

Let $\hat{e}_t$ be the residuals from the regression of $y_t$ on the deterministic trend $x_t$ and $\hat{\omega}_v^2$ be a consistent estimator of the long run variance of $v_t$, $\omega_v^2 = \Psi(1)^{-2} \sigma_w^2$, then the *LM* statistic can be constructed as follows:

$$LM = \frac{1}{T^2} \frac{\sum S_t^2}{\hat{\omega}_v^2} \,, \tag{4}$$

where $S_t$ is the partial sum process of the residuals $\sum_{j=1}^t \hat{e}_j$. Under the null, the *LM* statistic converges to $\int_0^1 V_X^2$, where $V_X(r) = W(r) - \left(\int_0^r X'\right)\left(\int_0^1 XX'\right)^{-1}\left(\int_0^1 X dW\right)$ is a generalized Brownian bridge process. When $x_t$ has a constant element, the process $V_X(r)$ is tied down to the origin at the ends of the [0,1] interval just like a Brownian bridge. In the case that $x_t$ is a constant, $V_X(r) = W(r) - rW(1)$ is a standard Brownian bridge. If $x_t$ is a linear trend, i.e. $x_t = (1, t)'$,

$$V_X(r) = [W(r) - rW(1)] + 6r(1-r)\left[\frac{1}{2}W(1) - \int_0^1 W(s)ds\right], \tag{5}$$

which is the sum of a standard Brownian bridge plus another factor $6r(1-r)\left[\frac{1}{2}W(1) - \int_0^1 W(s)ds\right]$, brought by the addition of a time trend $t$. This process is usually called a second-level Brownian bridge.

Kwiatkowski *et al.* (1992) present some simulation results illustrating the test size and power properties of this test. Several other studies have also investigated the small sample performance of the KPSS test, e.g., [17], [18], [29], and [30]. Shin [32] applied the same idea to a cointegrated time series and constructed a residual based test for the null hypothesis of cointegration. The focus of this paper is to study the small sample properties of the KPSS test for Stationarity and the Shin test for cointegration, and further, to study the applicability of the bootstrap method in testing the null of cointegration in small samples.

## 3. A Bootstrap Based Test for Stationarity

In order to bootstrap hypothesis testing, we need to construct the null distribution of the test statistic by resampling from the data. Thus the bootstrap pseudo-data should be generated from the model using the restricted estimator under the null hypothesis of cointegration. If the limit of the conditional distribution of the bootstrap statistic converges to $\int_0^1 V_x^2$, the bootstrap is asymptotically valid.

We follow the bootstrap procedures to generate bootstrap samples. If we denote the least squares residuals of

$$y_t = \gamma' x_t + v_t \tag{6}$$

as $\hat{v}_t$, i.e. $\hat{v}_t = y_t - \hat{\gamma}' x_t$, where $\hat{\gamma}$ is the OLS regression estimator, we can estimate the autoregressive coefficients $\Gamma_j$ by autoregressions on $\hat{v}_t$ as that in Section 2, and obtain

$$\hat{w}_t = \hat{v}_t - \hat{\Gamma}_1 \hat{v}_{t-1} - \Lambda - \hat{\Gamma}_q \hat{v}_{t-q} \tag{7}$$

We then center $\hat{w}_t$ at its sample average $\sum_{t=q+1}^{T} \hat{\varepsilon}_t / (T-q)$, denote the centered innovations as $\tilde{w}_t$ ($t = q + 1, \ldots, T$). The bootstrap innovations can then be generated. Draw an iid bootstrap sample $w_t^*$ from $\{\tilde{w}_t\}$. Generate the bootstrap residual process $v_t^*$ by

$$v_t^* = \hat{\Gamma}_1 v_{t-1}^* + \Lambda + \hat{\Gamma}_q v_{t-q}^* + w_t^*, \ t = q + 1, \ldots, T, \tag{8}$$

with $v_j^* = v_j$, for $j = 1, \ldots, q$, and generate the bootstrap sample $\{y_t^*\}$ by

$$y_t^* = \hat{\gamma}' x_t + v_t^* \tag{9}$$

Denote the residuals from the regression of $y_t^*$ on the deterministic trend $x_t$ as $e_t^*$,

$$y_t^* = \gamma^{*\prime} x_t + e_t^* . \tag{10}$$

then the bootstrapped test for stationarity can be constructed as follows:

$$LM^* = \frac{1}{T^2} \frac{\Sigma S_t^{*2}}{\hat{\omega}_v^{*2}} \tag{11}$$

where $S_t^*$ is the partial sum process of the residuals $\sum_{j=1}^{t} e_j^*$ .

The distribution of $LM^*$ is used to provide critical values for $LM$. Reject the null of cointegration at an $\alpha$ significance level if $LM > LM^*_{(1-\alpha)}$ where $LM^*_{(1-\alpha)}$ is the (1 - $\alpha$)-th percentile of the distribution of $LM^*$ .

The bootstrap approximation to the null distribution of the cointegration test is asymptotically valid if the limit of the conditional bootstrap distribution of $LM^*$ is the same as that given in Section 2. The asymptotic validity of the bootstrap procedure can be obtained similarly as in [21]. The proof is based on the establishment of the bootstrap invariance principles. Let $S_{[Tr]}$ and $S^*_{[Tr]}$ denote the standardized partial sum of $w_t$ and $w_t^*$ respectively, i.e., $S_{[Tr]} = T^{-1/2}\sum_{t=1}^{Tr} w_t$ , and $S^*_{[Tr]} = T^{-1/2}\sum_{t=1}^{Tr} w_t^*$ . Then by the invariance principle $S_{[Tr]} = T^{-1/2}\sum_{t=1}^{Tr} w_t \Rightarrow B_w(r)$ . We can show that the bootstrap invariance principle also holds, i.e., $S^*_{[Tr]} = T^{-1/2}\sum_{t=1}^{Tr} w_t^* \Rightarrow B_w(r)$ . Notice that $\hat{\Gamma}_j$ are $\sqrt{T}$ -consistent estimators of $\Gamma_j$, and $v_t^* = \hat{\Gamma}_1 v_{t-1}^* + \Lambda + \hat{\Gamma}_q v_{t-q}^* + w_t^*$ , thus $T^{-1/2}\sum_{t=1}^{Tr} v_t^* \Rightarrow B_v(r)$ Therefore the bootstrap is asymptotically valid. We only state the mail results without proof here.

**Theorem 1:** Under the null hypothesis and conditional on the data and for almost all sample paths

$$LM^* \Rightarrow \int_0^1 v_x^2 . \tag{12}$$

## 4. A Bootstrap Based Test for Cointegration

The Shin [32] test is essentially an extension of the KPSS test by adding $I(1)$ regressors to the KPSS components model. Consider the following cointegration regression:

$$y_t = Z_t' \beta + X_t ,\ t = 1, \ldots, T \tag{13}$$

where

$$X_t = \gamma_t + v_{2t} \tag{14}$$

$$\Delta Z_t = v_{2t} \tag{15}$$

$$\gamma_t = \gamma_{t-1} + u_t,\ \gamma_0 = 0,\ u_t \sim iid(0, \sigma_u^2) \tag{16}$$

and the scalar $v_{1t}$ and m-vector $v_{2t}$ are assumed to be strictly stationary and ergodic with zero mean, finite variance. We assume that the partial sum processes of $v_{1t}$ and $v_{2t}$ satisfy the invariance principles that $T^{-1/2}\sum_{t=1}^{[Tr]} v_{1t} \Rightarrow B_1(r)$, $T^{-1/2}\sum_{t=1}^{[Tr]} v_{2t} \Rightarrow B_2(r)$, where $B_1(r)$ and $B_2(r)$ are independent Brownian motions with variance $w_1^2$ and $\Omega_2$ respectively. We are interested in the null hypothesis that regression (13) is cointegrated, i.e.,

$$H_0 : \sigma_u^2 = 0\,. \tag{17}$$

Under the null hypothesis, $\gamma_t = 0$, $X_t = v_{1t}$. Thus $X_t$ is *I*(0) under the null hypothesis.

Let $\widehat{X}_t$ be the OLS residuals from the cointegrating regression (13). The partial sum process of $\widehat{X}_t$ is defined as $S_t$. The long-run variance of the regression error under the null is $s_l^2$, which can be consistently estimated by the nonparametric methods. The test statistics for the null of cointegration is given as:

$$CI = \frac{1}{T^2}\frac{\Sigma S_t^2}{s_l^2} \tag{18}$$

which has the following limiting distribution:

$$CI \to \int_0^1 Q^2 \tag{19}$$

where

$$Q = W_1 - \left(\int_0^r W_2'\right)\left(\int_0^1 W_2 W'\right)^{-1}\left(\int_0^1 W_2 dW_1\right) \tag{20}$$

and $W_1$ and $W_2$ are standard Brownian motions and are independent with each other.

The regression model (13) can be extended to include a constant term or a constant term and a time trend, in which case *Q* takes different forms. For detailed discussion of the asymptotic distribution of the cointegration test statistics, see [32].

In the above model we assume that there is no correlation between $v_{1t}$ and $v_{2t}$. In this case the limiting distribution of the cointegration parameter estimator is nuisance parameter free. Although this is rather restrictive, it simplifies our simulation which will be discussed in the following sections. Shin [32] considers different cases of exogenous or endogenous regressors. In the latter case a modified single equation model is used to correct for the endogeneity problem. For models without the strict exogenous assumption, other efficient cointegration regression estimation procedures can be used.

First, regression (13) is estimated to obtain the OLS parameter estimates $\hat{\beta}$ and residuals $\hat{X}_t$. Under the null hypothesis, the stationary process $\hat{v}_{1t}$ is obtained and resampled to get the bootstrap residuals $v_{1t}^*$ For the regressors, $v_{2t}$ can be generated from $\Delta Z_t$ and the innovations $v_{2t}$ are resampled to generate the bootstrap sample $Z_t^*$. Using these bootstrap samples, we can re-estimate regression (13) and construct the corresponding cointegration test statistics *CI**. The distribution of *CI** is then used to provide critical values for $\hat{CI}$.

Note that the generation of $v_{1t}^*$ and $v_{2t}^*$ depends on whether the processes $v_{1t}$ and $v_{2t}$ are serially correlated. In the case that $v_{1t}$ and $v_{2t}$ are serially correlated, bootstrap methods that can capture the serial correlation in $v_{1t}$ and $v_{2t}$ (say, the recursive bootstrap, moving block bootstrap, or sieve bootstrap methods) should be used. For convenience of asymptotic analysis, we assume that the serial correlation in $v_{1t}$ and $v_{2t}$ are characterized by stationary $AR(k)$ and $AR(q)$ processes respectively and use the recursive bootstrap in resampling $v_{1t}$ and $v_{2t}$. Thus

$$\Psi(L)v_{1t} = \varepsilon_t \tag{21}$$

where $\{\varepsilon_t\}$ are iid random variates with mean zero and variance $\sigma_\varepsilon^2$, $\Psi(L)$ is a *k*-th order polynomial of the lag operator $L$ defined as

$$\Psi(L) = 1 - \Gamma_1 L - \Lambda - \Gamma_k L^k, \tag{22}$$

and

$$A(L)v_{2t} = w_t, \tag{23}$$

where $\{w_t\}$ are iid random variates with mean zero and variance $\sigma_w^2$, $A(L)$ is a *q*-th order polynomial of $L$ defined as

$$A(L) = 1 - \alpha_1 L - \Lambda - \alpha_q L^q. \tag{24}$$

The distributions of $\varepsilon_t$ and $w_t$ are unknown and independent with each other. Given the assumptions on $\varepsilon_t$ and $w_t$ the partial sum processes of $\varepsilon_t$ and $w_t$ satisfy the invariance principles $T^{-1/2}\sum_{t=1}^{Tr}\varepsilon_t \Rightarrow B_\varepsilon(\gamma)$ and $T^{-1/2}\sum_{t=1}^{Tr} w_t \Rightarrow B_w(r)$, where $B_\varepsilon(\gamma)$ and $B_w(\gamma)$ are Brownian motion with variance $\sigma_\varepsilon^2$ and $\Omega_w$. By (21) and (23) we have $B_1(r) = \Psi(1)^{-1}B_\varepsilon(r)$, $B_2(r) = A(1)^{-1}B_w(r)$, $w_1^2 = \Psi(1)^{-2}\sigma_\varepsilon^2$, $\Omega_2 = A(1)^{-1}\Omega_w A(1)^{-1}$.

We consider the following bootstrap procedures:

(1) Estimate (13) to obtain the OLS parameter estimates $\hat{\beta}$ and residuals $\hat{X}_t$. The cointegration test statistics is calculated as $\hat{CI}$.

(2) Under the null hypothesis, $\hat{X}_t = \hat{v}_{1t}$. Estimate the autoregressive coefficients $\Gamma_j$, $j$ = 1, ..., $k$ by

$$\hat{v}_{1t} = \hat{\Gamma}_1 \hat{v}_{1,t-1} + \ldots + \hat{\Gamma}_k \hat{v}_{1,t-k} + \hat{\varepsilon}_t \tag{25}$$

and obtain $\hat{\varepsilon}_t$. Centering $\hat{\varepsilon}_t$ at its sample average $\sum_{t=k+1}^{T} \hat{\varepsilon}_t /(T-k)$, and denote the centered innovations as $\tilde{\varepsilon}_t$, $t = k +$ 1, ..., $T$.

(3) For the $I$(1) variables $Z_t$, we have $\Delta Z_t = v_{2t}$. Estimate the autoregressive coefficients $\alpha_1, \ldots, \alpha_q$ by

$$\hat{v}_{2t} = \hat{\alpha}_1 \hat{v}_{2,t-1} + \ldots + \hat{\alpha}_q \hat{v}_{2,t-q} + \hat{w}_t , \tag{26}$$

and obtain $\hat{w}_t$. Centering $\hat{w}_t$ at its sample average $\sum_{t=q+1}^{T} \hat{w}_t /(T-q)$, and denote the centered innovations as $\tilde{w}_t$, $t = q$ + 1, ..., $T$.

(4) Draw an iid bootstrap sample $\varepsilon_t^*$ from $\{\tilde{\varepsilon}_t\}$, and independently draw $w_t^*$ from $\{\tilde{w}_t\}$. Generate the bootstrap residual process $v_{1t}^*$ by

$$\hat{v}_{1t}^* = \hat{\Gamma}_1 \hat{v}_{1,t-1}^* + \ldots + \hat{\Gamma}_k \hat{v}_{1,t-k}^* + \varepsilon_t^* , t = k + 1, \ldots, T, \tag{27}$$

with $v_{1,j}^* = v_{1,j}$, for $j$ = 1, ..., $k$, and

$$\hat{v}_{2t}^* = \hat{\alpha}_1 \hat{v}_{2,t-1}^* + \ldots + \hat{\alpha}_k \hat{v}_{2,t-k}^* + w_t^* , t = q + 1, \ldots, T, \tag{28}$$

with $v_{2,j}^* = v_{2,j}$, for $j$ = 1, ..., $q$, and generate the bootstrap sample $\{Z_t^*\}$ by

$$Z_t^* = Z_{t-1}^* + v_{2t}^* , t = 2, \ldots, T, \tag{29}$$

with $Z_1^* = Z_1$.

(5) The bootstrap sample is generated by $y_t^* = Z_t^{*\prime} \hat{\beta} + X_t^*$. Regression (13) is re-estimated using the bootstrap data to get

the cointegration test statistics $CI^*$.

(6) The distribution of $CI^*$ is used to provide critical values for $\hat{CI}$. Reject the null of cointegration at an $\alpha$ significance level if $\hat{CI} > CI^*_{(1-\alpha)}$, where $CI^*_{(1-\alpha)}$ is the $(1-\alpha)$-th percentile of the distribution of $CI^*$.

Asymptotic validity of the bootstrap procedure can also be shown in a similar way as those in [21]. Notice that $\hat{v}_{1t} = y_t - Z_t'(\hat{\beta} - \beta) = y_t - Z_t' A_T$, where $A_T = O_p(T^{-1})$, similar results for the partial sums of $w_t^*$ and $\varepsilon_t^*$ can be proved,

$$T^{-1/2}\sum_{t=1}^{Tr} w_t^* \Rightarrow B_w(r),\ T^{-1/2}\sum_{t=1}^{Tr} \varepsilon_t^* \Rightarrow B_\varepsilon(r),\ T^{-1/2}\sum_{t=1}^{Tr} v_{1t}^* \Rightarrow B_1(r),\ T^{-1/2}\sum_{t=1}^{Tr} v_{2t}^* \Rightarrow B_2(r).$$

The asymptotic validity of the bootstrap is summarized without proof in the following Theorem.

**Theorem 2:** Under the null hypothesis and conditional on the data and for almost all sample paths,

$$CI^* \Rightarrow \int_0^1 Q^2 . \tag{30}$$

Notice that although the second order refinements can be shown in stationary time series regressions, second order improvements have not so far been proved in bootstrapped nonstationary time series regressions. This is largely because of the difficulty in developing valid high order extensions of the underlying functional central limiting theory on which the nonstationary regression asymptotics typically depend. Instead, we consider the asymptotic validity for bootstrapped nonstationary time series models.

## 5. Simulation Results

We first consider the KPSS test for stationarity in small samples. The model under investigation is regression (1) with a linear time trend. The null hypothesis is that $y_t$ is stationary or $I(0)$, i.e., $H_0: \sigma_u^2 = 0$. This implies that the time series $y_t$ under investigation can be decomposed into a linear time trend and a stationary component.

Various approaches are considered to study the small sample behavior of the KPSS stationarity test. It is known that the small sample size and power properties of the KPSS stationarity test are not satisfactory. In this study, we will focus on the bootstrap method. Since the recursive bootstrap discussed in the previous sections does not work well in the case of serial correlation and in particular when the order of serial correlation is unknown. Therefore we consider the stationary bootstrap. See [20] for details. The simulation results for the KPSS stationarity test are reported in Table 1.

**Table 1. Comparison of the KPSS test and the bootstrap test**

| | | | **5%** | | | **10%** | | |
|---|---|---|---|---|---|---|---|---|
| $\sigma_u^2$ | **AR(1)** | **Estimation** | ***l0*** | ***l4*** | ***l12*** | ***l0*** | ***l4*** | ***l12*** |
| 0 | 0.8 | KPSS | 89.12 | 33.58 | 6.93 | 94.42 | 50.51 | 30.28 |
| | | Bootstrap (10) | 22.10 | 13.96 | 5.32 | 36.26 | 25.76 | 12.12 |
| | | Bootstrap (30) | 41.78 | 21.42 | 6.38 | 58.82 | 33.16 | 13.36 |
| | 0 | KPSS | 5.41 | 4.26 | 4.46 | 10.57 | 10.47 | 26.01 |
| | | Bootstrap (10) | 4.38 | 4.82 | 4.78 | 10.14 | 10.72 | 9.58 |
| | | Bootstrap (30) | 3.84 | 5.22 | 4.84 | 9.56 | 10.76 | 9.62 |
| | -0.8 | KPSS | 0.00 | 1.35 | 3.63 | 0.00 | 5.44 | 26.03 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | Bootstrap (10) | 0.18 | 5.14 | 4.22 | 1.16 | 12.00 | 8.90 |
| | | Bootstrap (30) | 0.02 | 5.38 | 5.38 | 0.10 | 11.94 | 9.88 |
| 100 | 0.8 | KPSS | 97.25 | 62.72 | 17.79 | 98.79 | 76.70 | 48.39 |
| | | Bootstrap (10) | 44.28 | 34.34 | 14.80 | 58.66 | 47.46 | 26.60 |
| | | Bootstrap (30) | 69.94 | 47.20 | 16.00 | 82.00 | 59.36 | 27.60 |
| | 0 | KPSS | 97.28 | 63.20 | 18.29 | 98.77 | 76.86 | 48.76 |
| | | Bootstrap (10) | 45.36 | 34.52 | 14.96 | 58.74 | 48.28 | 26.34 |
| | | Bootstrap (30) | 69.58 | 46.10 | 16.68 | 82.34 | 58.70 | 27.74 |
| | -0.8 | KPSS | 97.10 | 63.25 | 18.19 | 98.52 | 77.00 | 49.09 |
| | | Bootstrap (10) | 44.88 | 34.92 | 15.42 | 59.20 | 48.44 | 27.14 |
| | | Bootstrap (30) | 70.26 | 47.62 | 17.44 | 82.12 | 60.38 | 28.02 |

We first estimate the model with the KPSS approach. The results are shown in the first lines of each panel in Table 1 indicated by KPSS. Then the bootstrap results are reported in the second and the third lines of each panel indicated by Bootstrap (10) and Bootstrap (30). The numbers 10 and 30 are the parameters controlling the block lengths in stationary bootstrap. Similar to KPSS, we estimate the long-run variance $\hat{\omega}_v^2$ (a consistent estimator of the long run variance of $v_t$) by considering three values of the number of lags ($l = l0$, $l4$, and $l12$). The stationary component of the regression residuals $v_t$ is assumed to be a stationary $AR(1)$ process with autoregressive parameter being $\{0.8, 0.0, -0.8\}$. For KPSS test, the critical values are taken from KPSS [16] which are based on a sample size of 20,000. In our simulation, a total of 5000 samples are generated for each parameter combination; for each sample, 200 bootstrap samples are generated. The test sizes are recorded for the 5% and 10% nominal levels. The results show that the KPSS approach has serious size distortions when $\sigma_u^2 = 0$. However, the bootstrap approach, although can not eliminate the size distortion completely, has significantly reduced the size distortion. When $\sigma_u^2 = 100$, the power properties of the two approaches can be compared. Note that the bootstrap approach also has reasonable rejection rates.

For the cointegration test of regression (13), the stationary component of the regression residuals $v_t$ is assumed to be a stationary $AR(1)$ process with autoregressive parameter being $\{0.0, 0.5, 0.8\}$. The OLS estimator is used because under the null of cointegration it is $T$-consistent. The regression error variance is set to 1. The number of regressors is set to 3. The regression parameters are all set to 1 without loss of generality. The innovations of $Z_t$ are independent of each other. In particular, they are independent to the regression residuals. Therefore the regressors are strictly exogenous. The test size will be studied under the null hypothesis. Then we examine the power of the test. The sample size is set to 50.

Regression (1) is estimated with a constant term and a time trend. We compare the Shin test and the bootstrap test for the null of cointegration. For Shin's test, the critical values are taken from Shin [32] which are based on a sample size of 2000. In our simulation, a total of 5000 samples are generated for each parameter combination; for each sample, 500 bootstrap samples are generated. The test sizes are recorded for the 5% and 10% nominal levels.

Since the regression error process is serially correlated, we use the stationary bootstrap method, which is described in [20]. The test size is the rejection rate of the null hypothesis when $\sigma_u^2 = 0$. The results in Table 2 show that Shin's test has considerable size distortions. On the other hand, the bootstrap test has smaller size distortion. When $\sigma_u^2 = 1, 100$, Shin's test has a larger rejection rate. However, this is because the test power is not size distortion adjusted. Therefore, overall the bootstrap test improves on the Shin test.

**Table 2. Comparison of the Shin test and the bootstrap test**

| $\sigma_u^2$ | AR(1) | Shin Test 5% | Shin Test 10% | Bootstrap test 5% | Bootstrap test 10% |
|---|---|---|---|---|---|
| 0.0 | 0.0 | 10.60 | 21.58 | 6.28 | 13.20 |
| | 0.5 | 17.90 | 33.70 | 9.58 | 17.58 |
| | 0.8 | 37.82 | 57.62 | 17.76 | 29.98 |
| 1.0 | 0.0 | 54.44 | 71.00 | 31.62 | 46.94 |
| | 0.5 | 50.72 | 68.46 | 28.76 | 44.22 |
| | 0.8 | 52.00 | 69.26 | 27.70 | 43.10 |
| 100 | 0.0 | 61.08 | 76.80 | 34.08 | 50.52 |
| | 0.5 | 58.92 | 75.60 | 32.96 | 48.82 |
| | 0.8 | 59.46 | 76.04 | 33.92 | 49.60 |

## 6. Conclusions

In this paper, we investigate the small sample properties of the KPSS test for stationarity and the Shin test for the null of cointegration. We also consider bootstrap-based tests as an alternative. Our simulation results show that the asymptotic approximations do not perform satisfactory in small to moderate samples. However, for the bootstrap approach we considered, it gives better small sample results.

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