Characterization of Partial Redundant Systems Using Joint Reliability Importance

H. Y. Koo[1)], J. S. Hong[2)], J. S. Lee[1)] and C. H. Lie[1)]

[1)]Department of Industrial Engineering, Seoul National University(chl@cybernet.snu.ac.kr)

[2)]Department of Industrial Engineering, Seoul National University of Technology

(hong@duck.snut.ac.kr)

ABSTRACT

We analyze the joint reliability importance of partial redundant systems comprised of independent components. This paper consists of two parts: One is to deal with the joint reliability importance of a partial redundant system (k-out-of-n system) which is most widely used in practice and the other is to investigate how to use the joint reliability importance in comparing various k-out-of-n systems. Firstly, we show that the joint reliability importance of k-out-of-n system with identically-independently-distributed components has a simple closed form and the sign of the joint reliability importance of any two components in k-out-of-n system with non-identical components is non-positive (nonnegative) if the reliabilities of all components are equal to or greater (less) than (k-1)/(n-1). Some comparisons of the joint reliability importance are depicted and expressed for interpreting the meaning of the joint reliability importance with respect to the level of redundancy. The result of this paper can be applied to determine the priority of preventive maintenance of components and can be helpful to decide the redundancy level of a partial redundant system in the system design phase.

1. Introduction

The joint reliability importance provides additional information, which the traditional marginal reliability importance such as Birnbaum importance cannot provide, to system designers [3, 4]. In this respect, the joint reliability importance is more useful than the marginal reliability importance since it indicates how components interact in determining the system reliability [1]. In order to explain the interactions between components, the concepts of reliability substitutes and reliability complements can be introduced [3]. Two components are called reliability substitutes (complements) if their sign of the joint reliability importance are non-positive (nonnegative). Reliability substitutes (complements) have the property that one component becomes more important when the other is failed (functioning). Therefore, the sign of the joint reliability importance plays an important role in the system reliability analysis such as preventive maintenance.

Two simple systems, series and parallel, have the simple property with respect to their sign of the joint reliability importance: it is nonnegative (non-positive) in series (parallel) system, that is, a series (parallel) system consists of reliability complements (substitutes). Although much has been written about the marginal reliability importance of special systems [6], little research is given to the joint reliability importance of special systems except the simple series and parallel systems. Thus, we analyze the characteristics of k-out-of-n systems by using the joint reliability importance.

In this paper, we show that the joint reliability importance of k-out-of-n system with identically-independently-distributed components has a simple closed form and that the sign of the joint reliability importance of any two components in k-out-of-n system with non-identical components is non-positive (nonnegative) if the reliabilities of all components are equal to or greater (less) than (k-1)/(n-1). As for the case of the non-identical components, Boland & Proschan firstly obtained the point (k-1)/(n-1) in order to show the Schur-convexity and concavity properties of k-out-of-n system [2]. Some comparisons of the joint reliability importance are depicted and expressed for interpreting the meaning of the joint reliability importance with respect to the level of redundancy. The result of this paper can be applied to determine the priority of preventive maintenance of components and can be helpful to decide the redundancy level of a partial redundant system in the system design phase.

Notations & Nomenclature

X_i	indicator for component i ; X_i =1 (0) if i works(fails)
p_i, q_i	$\Pr(X_i=1)$, $1-p_i$
$\boldsymbol{X}$	$(X_1, \ldots, X_n)$: state vector of components
$\boldsymbol{p}$	$(p_1, \ldots, p_n)$: probability vector of components
$\phi(\boldsymbol{X})$	structure function of the system
k-ou-of-n system	the system is good if and only if at least k components of its n components are good

2 Joint reliability importance of k-out-of-n system

A case of k-out-of-n system with identical components is analyzed first. Assuming the statistical independence between components and defining the joint reliability importance of two components, the simple closed form for the joint reliability importance of k-out-of-n system with identically-independently-distributed components are obtained along with the reliability function of k-out-of-n system. For $n \geq 3$ and $2 \leq k \leq n$, the joint reliability importance of k-out-of-n system with identically independently distributed components is illustrated to have the simple form. The value of the reliability of a component, p, which determines the sign of the joint reliability importance is then obtained. If p is greater than $\frac{k-1}{n-1}$, then the sign of the joint reliability importance is negative and vice versa.

As for the system with non-identical components, a region of component reliability is obtained comparing with the Schur-convexity instead of the exact formula of the joint reliability importance. The resulting region enables us to simply determine the sign of the joint reliability importance of k-out-of-n system with independent components. Also, we discuss the implications and applications of the sign of the joint reliability importance in the view point of the maintenance and design of a k-out-of-n system.

2.1 Joint reliability importance of identically-independently-distributed components

Assuming the statistical independence between components, the joint reliability importance of two components can be rewritten as [4]

$$\mathrm{JRI}(i\ , j) \equiv R(1_i, 1_j, \boldsymbol{p}) + R(0_i, 0_j, \boldsymbol{p}) - R(1_i, 0_j, \boldsymbol{p}) - R(0_i, 1_j, \boldsymbol{p}). \tag{1}$$

Theorem 1 gives the simple closed form for the joint reliability importance of k-out-of-n system with identically-independently-distributed components. It is easy to obtain the formula of the joint reliability importance by applying (1) to the reliability function of k-out-of-n system with identically-independently-distributed components.

Theorem 1. Let $n \geq 3$ and k be such that $2 \leq k \leq n$. The joint reliability importance of k-out-of-n system with identically-independently-distributed components has the following simple form.

$$\mathrm{JRI}(i\ , j) = p^{k-2}q^{n-k-1}\left[\binom{n-2}{k-2}-\binom{n-1}{k-1}p\right],\ \ \forall i\ \&\ j. \tag{2}$$

Proof) See [5].

As shown in (2), the joint reliability importance can be computed with ease in case of identically-independently-distributed components. Furthermore, the variations of the values of the joint reliability importance with regard to the component reliability can be compared with respect to the combinations of n & k as well. Some comparisons of the variations and properties for the characterization of k-out-of-n systems are investigated in section 3.

Using theorem 1, the following corollary can be obtained. Corollary 1 gives the value of p which determines the sign of the joint reliability importance.

Corollary 1. In k-out-of-n system with identically-independently-distributed components (n ≥ 3, k ≥ 2),

(a) JRI= 0, if $p = \frac{k-1}{n-1}$,

(b) JRI < 0, if $\frac{k-1}{n-1} < p < 1$,

(c) JRI > 0, if $0 < p < \frac{k-1}{n-1}$.

Proof) See [5].

Hence, we can easily find out whether the components of the system are reliability substitutes or complements if the reliability of the identically-independently-distributed component is given. In other words, optimum combinations of n & k can be easily obtained by using corollary 1 given the reliability of component, p.

2.2 Joint reliability importance of non-identical components

In case of k-out-of-n system with non-identical components, the reliability of this system is expressed by

$$R(\boldsymbol{p}\,;k,n) = \sum_{X_1+\Lambda+Xn\geq k} \prod_{i=1}^{n} p_i^{\,Xi} \prod_{i=1}^{n} (1-p_i)^{1-Xi} .$$

Therefore, JRI (i, j) is not obtained as a simple closed form like that of identically-independently-distributed case. But the sign of the joint reliability importance is determined by the values of p_i's, which is firstly obtained by Boland & Proschan. It is also shown that the reliability function is Schur-convex (concave) if $p_r \geq \frac{k-1}{n-1}$ ($p_r \leq \frac{k-1}{n-1}$) for all r [2]. The result of [2] enables us to prove theorem 2 that gives the point of values determining the sign of the JRI.

Theorem 2. In k-out-of-n system with non-identical components (n ≥ 3, k ≥ 2),

(a) JRI (i , j) ≥ 0 if $p_r \leq \frac{k-1}{n-1}$, $\forall r = 1,\Lambda\ ,n$,

(b) JRI $(i, j) \le 0$ if $p_r \ge \frac{k-1}{n-1}$, $\forall r = 1, \ldots, n$.

Proof) See [5].

Theorem 2 implies that the components of k-out-of-n system become reliability complements (substitutes) when $p_r \le \frac{k-1}{n-1}$ ($p_r \ge \frac{k-1}{n-1}$) for all r = 1, …, n . Also, it is implied in theorem 2 that in case of k-out-of-n system, Schur-convexity (concavity) of the system reliability function has the same meaning as the non-positivity (non-negativity) of the value of JRI has. In detail, if the component reliabilities of k-out-of-n system, p_r 's, are in the region of [(k-1)/(n-1), 1] for all r = 1,…, n, then we can say that (i) the reliability function is Schur-convex, (ii) more reliable component is more important in the view point of contributing to system reliability [2], (iii) the value of JRI is non-positive, thus, all the components are reliability substitutes and (iv) if preventive maintenance needs to be carried out, then the sequential or non-simultaneous maintenance should be done in an increasing order of component reliabilities as in the case of a parallel system. Similarly, if the component reliabilities are in the region of [0, (k-1)/(n-1)], then we can be sure that (i) the reliability function is Schur-concave, (ii) less reliable component is more important in the view point of contributing to system reliability, (iii) the value of JRI is non-negative, thus, all the components are reliability complements and (iv) the simultaneous preventive maintenance is preferred as in the case of a series system.

3 Comparisons of joint reliability importance in some k-out-of-n systems

Comparisons of joint reliability importance are made in two ways. First, the variations of joint reliability importance with respect to n and p are observed for fixed k (simply k=2). Second, the variations of joint reliability importance with respect to k and p are observed for fixed n (=2k).

Let us fix k (=2) in order to focus on the effect of redundancy for the joint reliability importance of k-out-of-n system. From (2) the joint reliability importance of k-out-of-n system with identically-independently-distributed components is

$$\text{JRI}\,(i, j) = q^{n-3}\left(1-(n-1)p\right).$$

Fig. 1 illustrates the joint reliability importance of 2-out-of-n systems. Fig. 1 shows that (i) the joint reliability importance of 2-out-of-n system converges to 0 as p increases for n≥4, (ii) for larger p and n≥4, the absolute value of the joint reliability importance is shown to decrease as n increases. Also parallel system has similar characteristics to 2-out-of-n system [5].

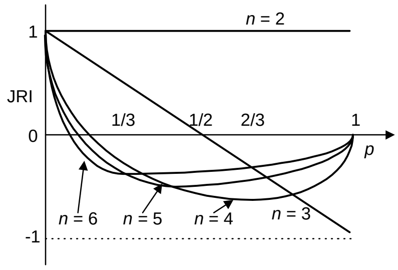

Fig. 1 Variations of joint reliability importance with respect to the level of redundancy

The value of the joint reliability importance represents the degree of interactions between two components with respect to the system reliability. From fig. 1, we can say that 2-out-of-n system has similar degree of interactions between two components to parallel system when components are highly reliable. It gives useful informations for determining the level of redundancy in 2-out-of-n system according to the value of p.

Using corollary 1, we can determine the value of n for the components to be reliability substitutes given the value of the component reliability, p. This means that if the value of n is less than $(1+p)/p$, then it is preferred to increase the level of redundancy, n, since the components are currently reliability complements as in the case of series system.

Let us denote the joint reliability importance of 2-out-of-n system with identically-independently-distributed components as JRI_n. Then, $JRI_n < JRI_{n+1}$, when $p > \frac{1}{2}$ [5]. This implies that the degree of interactions between components decreases as n increases if component reliability, *p*, is greater than 1/2. Thus, the interactive effect between components to the system reliability is decreasing when the size of the system, whose components are relatively reliable, is increasing.

To compare the k-out-of-n systems with the same level of redundancy, we let the number of redundant components to be equal to that of active components. Then, the system becomes k-out-of-2k system.

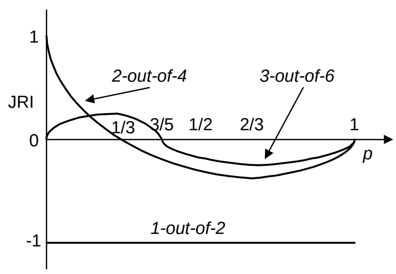

Fig. 2 Joint reliability importance of k-out-of-2k system with identically-independently-distributed components

Fig. 2 shows that the value of the joint reliability importance of the system decreases in ${}^1\!/_2 \le p \le 1$ as k increases. Using corollary 1, we can determine the value of n for the components to be reliability substitutes given the value of the component reliability, p. If the value of k is greater than $(1-p)/(1-2p)$ when $p < 1/2$, then the components are currently reliability complements as in the case of series system. Thus, in this case, decreasing the value of k to be less than $(1-p)/(1-2p)$ is preferable to increasing the size of the system in view of the interaction of components.

Suppose that JRI_k denotes the joint reliability importance of k-out-of-2k system with identically-independently-distributed components. Then, $JRI_k < JRI_{k+1}$ when $q < \frac{k}{6k-3}$ [5]. This implies that the degree of interactions between two components decreases as k and p increase and implies that k-out-of-2k system, as k and p increase, has less dependence between components with respect to the system reliability. This makes intuitive sense that k-out-of-2k system is more robust for dependent failures as k increases.

4 Conclusions

Joint reliability importance of partial redundant systems is analyzed assuming the components are statistically independent. In case of k-out-of-n system comprised of identically-independently-distributed components, an exact formula of the joint reliability importance is obtained. As for the non-identical components, the region of component reliability, which leads us to determine the sign of the joint reliability importance, is obtained. Some corollaries and propositions are also given for the characterization of k-out-of-n system in terms of the joint reliability importance. The results enable us to determine the sign of the joint reliability importance of k-out-of-n system with independent components. Hence, we can easily find out whether the components are reliability substitutes or complements if the reliabilities of components are in the region of $[(k-1)/(n-1), 1]^n$ or $[0, (k-1)/(n-1)]^n$. Given the sign of the joint reliability importance, we can rank the priority order of preventive maintenance task. In addition, the results in section 3 gives us an insight for choosing n or k in k-out-of-n system with regard to the interactions of components.

Acknowledgments

This research has been partially supported by the Brain Korea 21 project.

References

[1] Armstrong, M. J.: Joint reliability importance of components, IEEE Trans. Reliability, Vol. 44, pp 408-412, 1995

[2] Boland, P. J. & Proschan, F.: The reliability of K out of N systems, The Annals of Probability, Vol. 11, pp760-764, 1983

[3] Hagstrom, J. N.: Redundancy, substitutes and complements in system reliability, Tech. Report, College of Business Administration, University of Illinois, P.O. Box 4348, Chicago 60680, 1990.

[4] Hong, J. S. & Lie, C. H.: Joint reliability importance of two edges in an undirected network, IEEE Trans. Reliability, Vol. 42, pp17-23, 1993

[5] Koo, H. Y., Hong, J. S. & Lie, C. H.:JRI of k-out-of-n:G systems with independent components, working paper, Dep. of Industrial Engineering, Seoul National University, 2000

[6] Papastravridis, S.: The most important component in a consecutive-k-out-of-n: F system, IEEE Trans. Reliability, Vol. 36, pp266-267, 1987