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Decision Making under Ambiguous Risk

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Abstract

Ambiguous risk has been identified as a critical issue in decision making since the publication of Ellsberg's (1961) paradox, which demonstrated that typical choices made by subjects faced with ambiguous probabilities violate Savage's (1954) subjective expected utility theory. In this paper, we propose a generalized expected utility model based on the assumption that an individual first uses the mean value of the ambiguous probability to evaluate the alternative, and then considers the ambiguity effect on the preliminary evaluation using second-order beliefs. Ambiguity about probability is treated as a second order uncertainty modeled by a distribution. Then a decomposition approach is used to separate the effects of risk and ambiguity on decision evaluation. Our model is found useful in determining risk premium and "ambiguity premium" for pricing an alternative with ambiguous risk as often encountered in insurance decisions.

1. Introduction

Ambiguity has been identified as an important factor in insurance decision making (e.g., see Berger and Kunreuther, 1994; Hogarth and Kunreuther, 1989, 1992; Kunreuther, 1989). The insurance industry often must deal with the ambiguity associated with the probability of specific events occurring and/or the magnitude of potential losses. For example, ambiguity may be created by missing information, by concerns about source credibility, and by expert disagreement about probabilities and losses. For risks where there are considerable ambiguities on both probabilities and outcomes, the insurance industry is usually unwilling to extend coverage very widely or raises premiums dramatically. The notion of ambiguity is typically operationalized through a "second order" probability distribution on the probability of loss or on the parameters of the probability distribution of the loss.

Hogarth and Kunreuther (1992) conducted mail surveys and discovered ambiguity aversion in hypothetical decisions by both professional actuaries and experienced underwriters. These surveys used questionnaires with different scenarios: precise or ambiguous probabilities versus precise or ambiguous losses; they found that the actuaries and the underwriters add an "ambiguity premium" in pricing a given risk whenever probabilities and/or losses are ambiguous. Ambiguity is thus an important component in determining the market price for coverage. Hogarth and Kunreuther also found that the ambiguity of probability had more of an impact in raising premiums than the ambiguity of loss.

Additional empirical results on ambiguity in the insurance industry are provided by Hogarth and Kunreuther (1989) and Kunreuther (1989). Ambiguity is also an important factor in many other business decision problems (see Camerer and Weber, 1992, for a review).

Ambiguity has been a critical issue in the decision sciences since the publication of Ellsberg's (1961) paradox, which demonstrated that typical choices made by subjects faced with ambiguous probabilities violate Savage's (1954) subjective expected utility theory. Many theories and models have been proposed to explain the ambiguity effect on decision making since then. In a review paper, Camerer and Weber (1992) summarized four classes of theories on ambiguity as follows. Some theories are utility based: ambiguity about events lowers the utility of the consequences of those events (i.e., making utilities event-dependent). Other theories assume second-order beliefs to be weighted nonlinearly, which explains ambiguity aversion in much the same way that risk aversion is explained in expected utility models. A third class of theories assumes that there is a set of possible event probabilities and bases choice on the minimum subjective expected utility taken over all probabilities in the set. And a fourth class of theories either adjust an expected probability or weight it nonlinearly to reflect ambiguity, or allow probabilities that are nonadditive. In this short paper, we propose a model for decision making under ambiguous risks which includes the traditional subjective expected utility model as a special case. We also address some implications of our model in insurance decision making. In the present paper, we focus on the ambiguous probability of loss because it is the primary concern in the decision sciences (Camerer and Weber, 1992). The extension of our model for ambiguous losses should be a straight forward.

2. Expected Utility Models

The expected utility theory of von Neumann and Morgenstern (1947) and the subjective expected utility theory of Savage (1954) have been widely accepted as the leading normative theories of risky choice.The expected utility model can capture various risk attitudes of decision makers through different shapes of utility functions (Pratt, 1964). For example, a concave utility function implies that the decision maker is risk averse. The certainty equivalent (CE) of the lottery X can be obtained by solving $u(CE)$ = E[$u(X)$], where we denote *u(X)* as the utility of the probability distribution of the lottery, and the symbol E represents the expectation operator . If the inverse of a utility function exists, then we have

$$CE = u^{-1}\{\mathrm{E}[u(X)]\} \tag{1}$$

For example, if $u(x) = -be^{-cx}$, where b and c > 0, then its certainty equivalent form is as follows:

$$CE = \mu - \frac{1}{c}\log[\mathrm{E}(e^{-c(X-\mu)})] \tag{2}$$

where μ is the expected value of the lottery X and $\frac{1}{c}\log[\mathrm{E}(e^{-c(X-\mu)})]$ is called the risk premium. When the distribution of the lottery *X* is normal, then model (2) reduces to a mean-variance model as follows:

$$CE = \mu - \frac{1}{c}\sigma^2 \tag{3}$$

where σ^2 = E[$(X-\mu)^2$] is the variance. Note that the mean-variance model (3) is not an expected utility model, but is the certainty equivalent form of the expected exponential utility model when the distribution is normal.

When the distribution of *X* is not perfectly normal, an approximation for (2) can be obtained using the Taylor expansion for $\mathrm{E}[e^{-c(X-\mu)}] \approx 1 + c^2\sigma^2$. Then (2) becomes

$$CE = \mu - \frac{1}{c}\log(1 + c^2\sigma^2)\,. \tag{4}$$

When a decision maker is not very risk averse (i.e., c is very small), then (4) can be reduced to (3) (also by the Taylor expansion). Note that we can further incorporate skewness and other higher moments of the probability distribution into the model (4) if a quadratic approximation is not sufficient. Models (2), (3) and (4) imply a constant risk aversion since they are based on the exponential utility function.

For a simple insurance problem, if the loss of a possible claim is $x = -l$ and the probability of the claim is p, and if an insurer is constantly risk averse, then by (4) the certainty equivalent of insuring this possible claim for the insurer is

$$CE = -pl - \frac{1}{c}\log[1 + c^2 p(1-p)l^2] \tag{5}$$

Thus, a constant risk averse insurer will charge a higher premium ($pl + \frac{1}{c}\log[1 + c^2 p(1-p)l^2]$) than the expected loss (pl), even if administrative costs and profits are ignored Therefore, risk averse behavior in insurance decisions can be predicted by expected utility models.

But when there is ambiguity about the probability of a loss, these expected utility models are called into question. The subjective expected utility theory requires that decision probabilities are either objectively or subjectively known (Savage, 1954). Because expected utility models are linear in probabilities, they, in fact, imply neutrality toward ambiguity in the probability of loss (e.g., see the discussion in Raiffa, 1968). However, empirical studies show that individuals are usually ambiguity averse. In the insurance industry, ambiguity averse behavior raises questions about what models of choice insurers should utilize in making their premium recommendations.

3. A Decision Model under Ambiguous Risk

There are alternative concepts of ambiguity (see Camerer and Weber, 1992), but the strategy commonly used to represent probability uncertainty is a second order probability (SOP) distribution. This strategy is justified by the fact that in many instances the subjective probability of an event is naturally interpreted in this manner. For example, Ellsburg (1961) considered a choice between two urns, one with and unknown mix of 100 red and black balls, and the other with exactly 50 red and 50 black balls. It seems natural to assume that the probability of a red ball being drawn from the first urn would be interpreted as a SOP with equal probabilities of .01 assigned to the "first order" probabilities of 01, .02, .03,…, .99, 1.0. As a second example, Hogarth and Kunreuther (1992) defined an ambiguous probability to be one where the experts disagree on the chances of losses, so that some type of aggregation procedure is needed. This notion can also be modeled by a SOP over the opinions of the experts.Consider a simple lottery (or a random variable) $X = (x, p;\ 0, 1-p) \equiv (x, p)$, in which one has a chance to win (or to lose) x dollars with probability p, otherwise to win (or to lose) nothing (i.e., a zero outcome with probability 1-p). We denote $u(x)$ as the utility of outcome x with $u(0)$ = 0. Then the (subjective) expected utility model for evaluating the lottery X is:

$$\mathrm{E}[u(X)] = p\, u(x) \tag{6}$$

When ambiguity is involved in a probability estimate, the probability p in a lottery $X = (x, p)$ becomes a random variable, $\tilde{p}$, defined by a SOP $P(p)$ and with mean $\overline{p}$ and the lottery becomes $X = (x, \tilde{p})$. We define an ambiguity variable as $p' = \tilde{p} - \overline{p}$, which is also a SOP that is defined on differences from the mean $\overline{p}$. An ambiguous probability can be expressed as $\tilde{p} = (\overline{p} + \tilde{p} - \overline{p}) = (\overline{p} + p')$. For any unambiguous probability, p' = 0, and $\tilde{p} = \overline{p}$.

We decompose an ambiguous lottery into the following structure for decision evaluation: $X = (x, \tilde{p}) \Rightarrow (x, \overline{p}, p') \Rightarrow (X_{\overline{p}}, p')$, where $X_{\overline{p}} = (x, \overline{p})$ is just like the previous lottery with a "known" probability $\overline{p}$. This structure has an intuitive appeal; a decision maker first uses the expected value of the ambiguous probability to evaluate the lottery ($X_{\overline{p}}$), and then considers the ambiguity effect on the preliminary evaluation. The new representation $(X_{\overline{p}}, p')$ is a two-attribute structure. The second attribute arises because of ambiguity about the probability estimate, and provides a representation of the potential error that might be associated with the "best guess" for the probability of winning or losing, $\overline{p}$.

Let $>_{\mathbf{p}}$ be the binary preference relation for the structure for ambiguous lotteries. We assume the existence of a two-attribute expected utility model such that for two lotteries $(X_{\overline{p}}, p')$ and $(Y_{\overline{q}}, q')$,

$$(X_{\overline{p}}, p') >_{\mathbf{p}} (Y_{\overline{q}}, q') \iff \mathrm{E}[U(X_{\overline{p}}, p')] > \mathrm{E}[U(Y_{\overline{q}}, q')].$$

Here the expectation should be taken over the joint distribution of random variable $X_{\overline{p}}$ (or $Y_{\overline{q}}$) and random variable $\tilde{p}$ (or $\tilde{q}$). We assume that the two distributions are statistically independent; then $\mathrm{E}[U(X_{\overline{p}}, p')] = \mathrm{E}^P \mathrm{E}^X [U(X_{\overline{p}}, p')]$, where E^X represents the expectation taken over $X_{\overline{p}}$ and E^P the expectation over the second order probability distribution p'. We also require that the two attribute utility model be consistent with the traditional expected utility model when there is no ambiguity in the probability.

Furthermore, to obtain a separable form of the general model, we need to have an independence condition stated as follows.

Definition. Ambiguity independence is satisfied if there exists a $\overline{p} \in (0,1)$ for which $(X_{\overline{p}}, p') >_{\mathbf{p}} (X_{\overline{p}}, q')$, then $(X_{\overline{r}}, p') >_{\mathbf{p}} (X_{\overline{r}}, q')$ for any $\overline{r} \in (0, 1)$.

This condition means that the effect of ambiguity on preferences does not depend on a particular choice of unambiguous lotteries $X_{\overline{r}}$, where $\overline{r} \in (0, 1)$. The ambiguity independence condition is analogous to the utility independence condition for a multiattribute utility model (see Keeney and Raiffa, 1976, pp 224-229), which leads to a separable form of the general model.

THEOREM. The two attribute utility model can be decomposed into the following form,

$$\mathrm{E}[U(X_{\overline{p}}, p')] = \mathrm{E}^X[u(X_{\overline{p}})] - \mathrm{E}^P[\varphi(\tilde{p} - \overline{p})]\,\mathrm{E}^X[g(X_{\overline{p}})] \qquad (7)$$

if and only if the ambiguity independence condition holds, where $g(x) > 0$ and $\varphi(0) = 0$.
(The proof of this Theorem is provided in the appendix.)

Model (7) has an intuitive appeal: if there is no ambiguity associated with the probability, then this model reduces to the traditional expected utility model; but because of ambiguity, the expected utility is reduced by an amount proportional to $\mathrm{E}^X[g(X_{\bar{p}})]$ if an individual is ambiguity averse. This model can also capture different attitudes toward ambiguity: if $\mathrm{E}^p[\varphi(\tilde{p}-\bar{p})] > 0$, then the individual is ambiguity averse; if $\mathrm{E}^p[\varphi(\tilde{p}-\bar{p})] < 0$, then ambiguity seeking; and if $\mathrm{E}^p[\varphi(\tilde{p}-\bar{p})] = 0$, then ambiguity neutral. Empirical studies show that under some circumstances individuals may also be ambiguity seeking or ambiguity neutral (e.g., see Kahn and Sarin, 1988). It is not clear to us at this point how to determine the function $g(x)$. The only constraint for $g(x)$ is of positivity.

One interesting choice of the function $g(x)$ is to let $g(x) = -u(x) > 0$ (e.g., $u(x) = -be^{-cx}$, then $g(x) = be^{-cx}$); then model (7) becomes:

$$\mathrm{E}[U(X_{\bar{p}}, p')] = \mathrm{E}^X[u(X_{\bar{p}})]\,\mathrm{E}^p[\varphi^*(\tilde{p}-\bar{p})] \tag{8}$$

where $\mathrm{E}^p[\varphi^*(\tilde{p}-\bar{p})] = 1 + \mathrm{E}^p[\varphi(\tilde{p}-\bar{p})] > 0$ serves as an ambiguity discounting factor on the expected utility. If one is ambiguity averse, then $\mathrm{E}^p[\varphi^*(\tilde{p}-\bar{p})] < 1$.

4. Some Examples

By choosing some appropriate functional forms for u, φ and g in (7) and (8), we can obtain specific decision models. In this section, we illustrate some examples of those models and discuss their implications.

(1). Kahn and Sarin's (1988) model

For a simple lottery case $X_{\bar{p}} = (x, \bar{p})$, we let

$$\mathrm{E}^p[\varphi^*(\tilde{p}-\bar{p})] = 1 + \int_0^1 \frac{(p-\bar{p})}{\bar{p}} e^{-\lambda(p-\bar{p})/\tau} P(p)\,dp$$

where $P(p)$ is the second order probability distribution on the ambiguous probability, τ is its standard deviation, and λ is a constant. Then model (8) becomes:

$$\begin{aligned}\mathrm{E}[U(X_{\bar{p}}, p')] &= \bar{p}u(x)[1 + \int_0^1 \frac{(p-\bar{p})}{\bar{p}} e^{-\lambda(p-\bar{p})/\tau} P(p)dp] \\ &= u(x)[\bar{p} + \int_0^1 (p-\bar{p}) e^{-\lambda(p-\bar{p})/\tau} P(p)dp] \\ &= w(\tilde{p})u(x)\end{aligned} \tag{9}$$

where $w(\tilde{p}) = \bar{p} + \int_0^1 (p-\bar{p}) e^{-\lambda(p-\bar{p})/\tau} P(p)dp$ is a decision weight due to ambiguity. Model (9) was first proposed by Kahn and Sarin (1988), which is shown as a special case of our model. In Kahn and Sarin's (1988) paper, they did not provide the fundamental assumptions for their model. They constructed their model based on an intuitive approach.

(2) Boiney's (1993) model

Boiney (1993) proposed a model similar to Kahn and Sarin's (1988) from an empirical consideration. He employed the disappointment theory (Bell, 1985) to construct a decision weight as follows:

$$w(\tilde{p}) = \bar{p} + \int_0^1 A(p-\bar{p})P(p)dp \tag{10}$$

where $A(p-\bar{p}) = e(p-\bar{p})^2$ when $p > \bar{p}$; and $A(p-\bar{p}) = -d(p-\bar{p})^2$ when $p < \bar{p}$. By choosing the appropriate parameters e and d, this model can capture individuals' attitudes toward ambiguity and the asymmetric effects of the ambiguous probability on preference. Boiney's model is also a special case of our model.

(3) An exponential model under ambiguous risk

For model (8), if we choose $u(x) = -be^{-cx}$ and $\mathrm{E}^p[\varphi^*(\tilde{p}-\bar{p})] = \mathrm{E}^p[e^{-\lambda(\tilde{p}-\bar{p})}]$, then it becomes:

$$\begin{aligned}\mathrm{E}[U(X_{\bar{p}}, p')] &= \mathrm{E}^X(-be^{-cX_{\bar{p}}})\,\mathrm{E}^p[e^{-\lambda(\tilde{p}-\bar{p})}] \\ &= (-be^{-c\mu})\mathrm{E}^X[e^{-c(X_{\bar{p}}-\mu)}]\,\mathrm{E}^p[e^{-\lambda(\tilde{p}-\bar{p})}]\end{aligned} \tag{11}$$

where μ is the mean of the lottery $X_{\bar{p}}$. Model (11) shows that if the lottery had no risk and ambiguity, then an individual would have a utility, $-be^{-c\mu}$; but because it is risky, the utility is discounted by a risk factor $\mathrm{E}^X[e^{-c(X_{\bar{p}}-\mu)}]$; and also because there is ambiguity on the probability, the utility is further discounted by an ambiguity factor $\mathrm{E}^p[e^{-\lambda(\tilde{p}-\bar{p})}]$. The certainty equivalent form of model (11) can be obtained as follows:

$$CE = \mu - \frac{1}{c}\log\{\mathrm{E}^X[e^{-c(X_{\bar{p}}-\mu)}]\} - \frac{1}{c}\log\{\mathrm{E}^p[e^{-\lambda(\tilde{p}-\bar{p})}]\} \tag{12}$$

Compared with model (2) for no ambiguity case, this model has an extra term, $\frac{1}{c}\log\{\mathrm{E}^p[e^{-\lambda(\tilde{p}-\bar{p})}]$, which can be explained as the "ambiguity premium". This simple model can explain Hogarth and Kunreuther's (1992) empirical findings that actuaries and underwriters will add an "ambiguity premium" in pricing a given risk whenever probabilities are ambiguous.

(4) A mean-variance model under ambiguous risk

Mean-variance models have been the most common tool used in modeling financial and insurance problems. If we choose a linear function for the expected utility $\mathrm{E}^X[u(X_{\bar{p}})]$, a variance term for $\mathrm{E}^X[g(X_{\bar{p}})]$ (> 0) and $\frac{1}{c}\mathrm{E}^p[e^{-\lambda(\tilde{p}-\bar{p})}]$ for the ambiguity factor, then model (7) becomes:

$$\mathrm{E}[U(X_{\bar{p}}, p')] = \mu - \frac{1}{c}\mathrm{E}^{p}[e^{-\lambda(\tilde{p}-\bar{p})}]\sigma^{2} \tag{13}$$

Compared with the traditional mean-variance model (4), the tradeoff between mean and variance now depends on the ambiguity measure $\mathrm{E}^{p}[e^{-\lambda(\tilde{p}-\bar{p})}]$. If there is no ambiguity, model (13) just reduces to the traditional mean-variance model (3).

As we demonstrated, model (7) is very general, including some other models that have been proposed as special cases and leading to new functional forms of decision model under ambiguous risk. We expect that these models we have proposed in this paper can be used for both descriptive and prescriptive purposes in insurance decision making and other contexts.

5. Appendix

Proof of Theorem.

Let x be a realization of $X_{\bar{p}}$ and r be a realization of p'. For $(x, r) \in \mathrm{R}\times\mathrm{R}$, the two-attribute utility function $U(x,r)$ can be decomposed as follows

$$U(x,r) = f(x) + g(x)U(c,r)$$

if and only if the ambiguity independence condition holds (see Keeney and Raiffa, 1976, pp 224-229, for the utility independence that is analogous to our ambiguity independence condition), where c can be any constant and $g(x) > 0$. We choose $c = 0$, and let $\varphi(r) = -U(0,r)$ (i.e., ambiguity aversion). Taking the expectation for the above equation and using the statistical independence condition, we have

$$\begin{aligned} \mathrm{E}[U(X_{\bar{p}}, p')] &= \mathrm{E}[f(X_{\bar{p}})] - \mathrm{E}\{[\varphi(p')][g(X_{\bar{p}})]\} \\ &= \mathrm{E}^{X}[f(X_{\bar{p}})] - \mathrm{E}^{p}[\varphi(p')]\mathrm{E}^{X}[g(X_{\bar{p}})] \end{aligned}$$

We require that when there is no ambiguity, the two attribute utility model reduces to the traditional expected utility model. Choosing the scale such that $U(0,0) = 0$, then $\varphi(0) = 0$ (because $g(x) > 0$). Thus $\mathrm{E}[U(X_{\bar{p}}, 0)] = \mathrm{E}^{X}[f(X_{\bar{p}})] = \mathrm{E}^{X}[u(X_{\bar{p}})]$. Finally, we have our model as follows (note that $p' = \tilde{p} - \bar{p}$):

$$\mathrm{E}[U(X_{\bar{p}}, p')] = \mathrm{E}^{X}[u(X_{\bar{p}})] - \mathrm{E}^{p}[\varphi(\tilde{p} - \bar{p})]\,\mathrm{E}^{X}[g(X_{\bar{p}})].$$

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