COMPARISON OF NORMAL VARIANCE ESTIMATORS IN TERMS OF PITMAN NEARNESS CRITERION

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ABSTRACT

For estimating a normal variance under squared error loss function it is well known that the best affine (location and scale) equivariant estimator, which is better than the maximum likelihood estimator as well as the unbiased estimator, is also inadmissible. The improved estimators, e.g., Stein type, Brown type and Brewster-Zidek type, are all scale equivariant but not location invariant. Lately a good amount of research has been done to compare the improved estimators in terms of risk, but very little attention had been paid to compare these estimators in terms of Pitman nearness criterion. In this paper we have undertaken a comprehensive study to compare various variance estimators in terms of Pitman nearness criterion, which has long been over due, and have made some interesting observations in the process.

Keywords and phrases: Affine equivariance, loss function, risk function, non-central chi-square distribution

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1. INTRODUCTION

Assume that we have independent random observations X and S such that $X = (X_1, X_2, \cdots, X_p)$ follow a $N_p(q, s^2 I_p)$ (p-dimensional normal) distribution and (S/s^2) follow a c^2_{m-1} (chi-square with $(m-1)$ *d. f.*) distribution. Consider the problem of estimation of s^2 efficiently.

The above-described model is encountered if one has independent and identically distributed (*iid*) observations $X_1, X_2, \cdots, X_n$ from a $N_p(q, s^2 I_p)$ distribution. The data can be reduced by sufficiency principle, and one needs to focus only on $X = \sqrt{n}\,\overline{X}$, $\overline{X} = (\sum_{i=1}^{n} X_i / n)$ and $S = \sum_{i=1}^{n} \left\| X_i - \overline{X} \right\|^2$. Note that X follows $N_p(m, s^2 I_p)$ and $S/s^2 \sim c^2_{m-1}$ with $m = \sqrt{n}q$ and $(m-1) = p(n-1)$.

Similarly, in a linear model setup $Y_{n\,1} = X_{n\,p} b_{p\,1} + {}_{n\,1}$ where ${}_{n\,1}$ follows $N_n(0, s^2 I_n)$ distribution, let b be the least squares estimate of b and $M_{p\,p}$ (p. d.) be such that $M M = (X X)$, then (Mb) plays the role of X and S plays the role of error sum of squares (SSE) for suitable choices of q and m.

Even though this article deals with estimation of s^2, the techniques discussed here can be used, with suitable modification, for estimating s^{2a} for any $a > 0$. In particular, one can take $a = 1/2$ to estimate s, the normal standard deviation. The special case $p = 2$ of the above setup has many applications in defense research and development while estimating the accuracy of a weapon system. Accuracy of a weapon system is measured by CEP, called Circular Probable Error. For example, in a test range a new surface to surface missile in being tested and the location of the target is known, say, $O = (0, 0)$. From a fixed distance when missiles are test fixed, we observe the points of impact, i. e., bivariate locations $X_1, X_2, \cdots, X_n$ which are assumed to follow a bivariate normal distribution with mean $m = (m_1, m_2)$ and diagonal dispersion matrix $s^2 I_2$. When the system is 'in order' (or 'in focus'), the mean m is expected to be 0, the target location; and the system is 'out of focus' if m is different from O and usually unknown. The radius of the circular area centered at m which gives probability mass of 0.5 is called the CEP of the weapon system (particular type of missiles). It can be shown that CEP is proportional to s, and hence CEP estimation essentially boils down to estimation of s.

In classical statistics, usual estimators of s^2 are (i) the unique minimum variance unbiased estimator (UMVUE) given by

$$s^2_u = S/(m-1); \tag{1.1}$$

and (ii) the maximum likelihood estimator (MLE) given by

$$s^2_{ml} = S/(m+p-1). \tag{1.2}$$

In a decision-theoretic setup the two most commonly used loss functions are

$$L_S(s^2, w) = (s^2/s^2 - 1)^2 \tag{1.3}$$

$$L_E(s^2, w) = (s^2/s^2) - ln(s^2/s^2) - 1 \tag{1.4}$$

where s^2 is an estimator of s^2 and $w = (q, s^2)$. The loss functions L_S and L_E are called respectively the squared error loss (SEL) and the entropy loss (EL).

If we consider the group G_A of affine transformations (i.e., (X, S) $(aX + b, a^2 S)$, $a > 0$, b ${}^p = p-$dimensional real space), then the affine equivariant estimators have the form $s^2_c = cS$, where $c > 0$ is a constant. Since the group G_A (and the corresponding induced group $\overline{G}_A$ acting on $\Omega = \left\{ w = (q, s^2) \middle| q \;\; {}^p, s^2 > 0 \right\}$ such that (q, s^2) $(aq + b, a^2 s^2)$, $a > 0$, b ^{p}) is transitive, an affine equivariant estimator s^2_c has constant risk on Ω. Therefore, one can find the best affine equivariant estimator (BAEE) of s^2 by minimizing the risk of s^2_c with respect to (w. r. t.) c. The BAEEs of s^2 under L_S and L_E are respectively.

$$s^2_S = S/(m+1) \quad \text{and} \quad s^2_E = s^2_u = S/(m-1) \tag{1.5}$$

Interestingly, s^2_S (s^2_E) is inadmissible under L_S (L_E), and improved estimators are only scale equivariant but not location invariant. Stein (1964) showed that under L_S, an improved estimator of s^2 can be found as

$$s^2_{S(S)} = min\left\{ S/(m+1),\ (S + \|X\|^2)/(m+p+1) \right\}, \tag{1.6}$$

which is uniformly better than s^2_S. Brown (1968) proposed a similar but somewhat different estimator of s^2 under

L_S of the form

$$s^{\,2}_{S(B)} = \left\{ c_0 I(F < r_0) + (m+1)^{-1} I(F \quad r_0) \right\} S \,, \tag{1.7}$$

where $r_0 > 0$ is any constant, $F = \|X\|^2 \big/ S$ and $c_0 = c_0(r_0)$ is a suitable constant dependent on r_0, and $c_0 < (m+1)^{-1}$. However, both $s^{\,2}_{S(S)}$ and $s^{\,2}_{S(B)}$ are nonanalytic and hence inadmissible. Brown's technique was further extended by Brewster and Zidek (1974) who obtained an admissible improved estimator of $s^{\,2}$ as

$$s^{\,2}_{S(BZ)} = \left\{ 1 - f_{S(BZ)}(U) \right\} S \big/ (m+1) \,, \tag{1.8}$$

where $F = \|X\|^2 / S$, $U = (1+F)^{-1}$ and

$$f_{S(BZ)}(U) = \frac{2(m+p+1)^{-1}(1-U)^{p/2} U^{(m+1)/2}}{{}^{1}_{U}(1-y)^{p/2-1}\, y^{(m+1)/2} dy} \,. \tag{1.9}$$

For a comprehensive review on normal variance estimation and related topics see Pal, Ling and Lin (1998). Estimators analogous to (1.6), (1.7) and (1.8) under the loss L_E can be derived. Such estimators are mentioned in Section 3 for further analysis.

While emphasis had been given to compare various variance estimators in terms of risk, very little attention had been paid to do the same in terms of another important criterion namely, the Pitman nearness criterion (PNC).

Definition 1.1. Given two estimators, say $s^{\,2}_1$ and $s^{\,2}_2$, of $s^{\,2}$, $s^{\,2}_1$ is said to be better than $s^{\,2}_2$ in terms of PNC if

$$\Delta_w(s^{\,2}_1 \big| s^{\,2}_2) = P\left((s^{\,2}_1 - s^{\,2})^2 \quad (s^{\,2}_2 - s^{\,2})^2 \right) \quad 0.5$$

$$\text{i.e., } \Delta_w(s^{\,2}_1 \big| s^{\,2}_2) = P\left((s^{\,2}_1 \big/ s^{\,2} - 1)^2 \quad (s^{\,2}_2 \big/ s^{\,2} - 1)^2 \right) \quad 0.5 \,. \tag{1.10}$$

The major difference between the usual risk criterion and the PNC is that the latter is nontransitive i.e., if $s^{\,2}_1$ is better than $s^{\,2}_2$ in terms of PNC, and $s^{\,2}_2$ is better than $s^{\,2}_3$ in terms of PNC, then it does <u>not</u> imply that $s^{\,2}_1$ is better than $s^{\,2}_3$ in terms of PNC. As a result, comparison under PNC can sometimes be more complicated and a bit confusing.

The renewed interest in PNC was due to Rao (1981), who claimed that the PNC was more appropriate than the quadratic loss function for evaluating an estimator. For further general discussion on this topic one can see Rao, Keating and Mason (1986).

In Section 2 we first consider comparison of three affine equivariant estimators, e.g., $s^{\,2}_{ml}$, $s^{\,2}_u = s^{\,2}_E$ and $s^{\,2}_S$, in terms of PNC. It appears, quite interestingly, that the unbiased estimator emerges as the most preferable among the three affine equivariant estimators. In Section 3 we compare $s^{\,2}_u = s^{\,2}_E$ against $s^{\,2}_{E(S)}$ and $s^{\,2}_{E(BZ)}$, Stein type and Brewster-Zidek type improved estimators under L_E respectively. Interestingly it is found that both $s^{\,2}_{E(S)}$ and $s^{\,2}_{E(BZ)}$ are much inferior to $s^{\,2}_u$ (UMVUE) in terms of PNC. It turns out that $s^{\,2}_u$ is better (PNC) than other variance estimators and this is the main contribution of our investigation.

2. COMPARISON OF AFFINE EQUIVARIANT ESTIMATORS

Take two affine equivariant estimators $s^{\,2}_1 = c_1 S$ and $s^{\,2}_2 = c_2 S$. Then $s^{\,2}_1$ is closer to $s^{\,2}$ than $s^{\,2}_2$ with probability

$$\Delta_w(s^{\,2}_1 \big| s^{\,2}_2) = P_w\left((s^{\,2}_1 \big/ s^{\,2} - 1)^2 \quad (s^{\,2}_2 \big/ s^{\,2} - 1)^2 \right). \tag{2.1}$$

The probability $\Delta_w(s^{\,2}_1 \big| s^{\,2}_2)$ can be simplified further as

$$\Delta_w(s^{\,2}_1 \big| s^{\,2}_2) = P_w \quad \left(\frac{S}{s^{\,2}}\right) \left(c_1^2 - c_2^2\right)\left(\frac{S}{s^{\,2}}\right) - 2(c_1 - c_2) \quad 0$$

$$= \begin{array}{ll} P(S \big/ s^{\,2} \quad 2(c_1 + c_2)^{-1}) \,, & if \;\; c_1 > c_2 \\ P(S \big/ s^{\,2} \quad 2(c_1 + c_2)^{-1}) \,, & if \;\; c_1 < c_2 \,. \end{array} \tag{2.2}$$

Call $c^* = 2 \big/ (c_1 + c_2)$. Since $S \big/ s^{\,2}$ is a $c^{\,2}_{m-1}$ random variable we have

$$\Delta_w(s^{\,2}_1 \big| s^{\,2}_2) = \begin{array}{ll} P(c^{\,2}_{m-1} \quad c^*) \,, & if \;\; c_1 > c_2 \\ P(c^{\,2}_{m-1} \quad c^*) \,, & if \;\; c_1 < c_2 \,. \end{array} \tag{2.3}$$

Also, form the expression (2.1) it is conveniently noted that $\Delta_w(s_1^2 | s_2^2) = 1 - \Delta_w(s_2^2 | s_1^2)$.

2.1 Computation of Δ_w's

(1) Comparison of s_{ml}^2 and s_S^2.

Take $c_1 = (m+1)^{-1}$ and $c_2 = (m+p+1)^{-1}$. Then define $c_1^* = 2(m+1)(m+p+1)/(2m+p+2)$. Then from (2.3),

$$\Delta_w(s_S^2 | s_{ml}^2) = P(c_{m-1}^2 \quad c_1^*). \tag{2.4}$$

(2) Comparison of s_u^2 and s_{ml}^2.

Take $c_1 = (m-1)^{-1}$ and $c_2 = (m+p+1)^{-1}$. Define $c_2^* = 2(m-1)(m+p+1)/(2m+p)$. Form (2.3) it is readily seen that

$$\Delta_w(s_u^2 | s_{ml}^2) = P(c_{m-1}^2 \quad c_2^*). \tag{2.5}$$

(3) Comparison of s_u^2 and s_S^2.

Take $c_1 = (m-1)^{-1}$ and $c_2 = (m+1)^{-1}$. Also define $c_3^* = (m^2-1)/m$. Therefore, from (2.3) we have

$$\Delta_w(s_u^2 | s_S^2) = P(c_{m-1}^2 \quad c_3^*). \tag{2.6}$$

of m and p. Note that while $\Delta_w(s_S^2 | s_{ml}^2)$ and $\Delta_w(s_u^2 | s_{ml}^2)$ are dependent on both m and p, $\Delta_w(s_u^2 | s_S^2)$ is free from p.

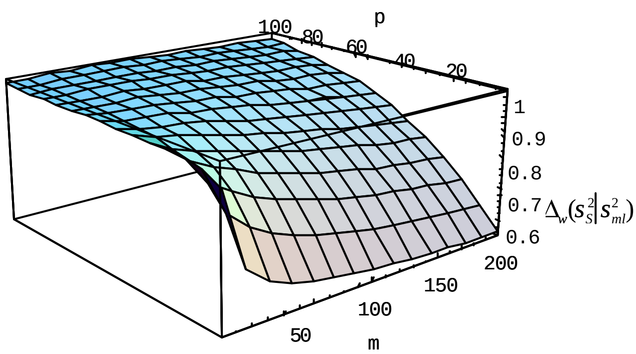

Figure2.1. 3-D graph of $\Delta_w(s_S^2 | s_{ml}^2)$.

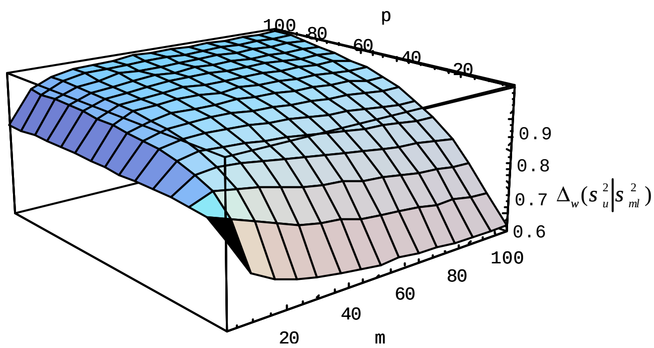

Figure2.2. 3-D graph of $\Delta_w(s_u^2 | s_{ml}^2)$.

Table 2.1. Values of $\Delta_w(s_S^2 | s_{ml}^2)$ for various m and p.

p \ m	2	3	4	5	10	15	20	25	50	100
1	0.9359	0.8916	0.8586	0.8328	0.7557	0.7153	0.6895	0.6712	0.6236	0.5884
2	0.9471	0.9093	0.8800	0.8564	0.7820	0.7407	0.7135	0.6938	0.6414	0.6017
3	0.9545	0.9216	0.8956	0.8743	0.8041	0.7631	0.7352	0.7147	0.6585	0.6147
4	0.9596	0.9305	0.9075	0.8883	0.8230	0.7829	0.7549	0.7339	0.6748	0.6274
5	0.9633	0.9373	0.9167	0.8994	0.8391	0.8005	0.7728	0.7516	0.6905	0.6399
6	0.9661	0.9426	0.9240	0.9084	0.8529	0.8612	0.7891	0.7680	0.7054	0.6520
7	0.9683	0.9468	0.9300	0.9158	0.8649	0.8301	0.8038	0.7830	0.7196	0.6639
8	0.9701	0.9502	0.9349	0.9220	0.8753	0.8426	0.8173	0.7969	0.7332	0.6754
9	0.9715	0.9531	0.9390	0.9272	0.8844	0.8537	0.8295	0.8098	0.7461	0.6866
10	0.9728	0.9554	0.9424	0.9317	0.8924	0.8637	0.8407	0.8216	0.7584	0.6975
20	0.9788	0.9676	0.9604	0.9551	0.9379	0.9245	0.9121	0.9001	0.8525	0.7905
30	0.9810	0.9721	0.9672	0.9641	0.9564	0.9508	0.9449	0.9388	0.9081	0.8571
40	0.9821	0.9744	0.9707	0.9688	0.9659	0.9643	0.9620	0.9592	0.9411	0.9032
50	0.9828	0.9759	0.9728	0.9716	0.9716	0.9722	0.9719	0.9710	0.9610	0.9344
100	0.9842	0.9788	0.9771	0.9772	0.9822	1.0000	1.0000	0.9903	0.9924	0.9897

Table 2.2. Values of $\Delta_w(s_u^2 | s_{ml}^2)$ for various m and p.

p \ m	2	3	4	5	10	15	20	25	50	100
1	0.7941	0.7603	0.7385	0.7219	0.6721	0.6457	0.6286	0.6163	0.5843	0.5603
2	0.8033	0.7769	0.7593	0.7452	0.6985	0.6711	0.6526	0.6390	0.6021	0.5737
3	0.8096	0.7889	0.7752	0.7636	0.7213	0.6940	0.6746	0.6601	0.6193	0.5868
4	0.8141	0.7981	0.7877	0.7785	0.7410	0.7145	0.6949	0.6798	0.6358	0.5996
5	0.8176	0.8053	0.7978	0.7907	0.7583	0.7330	0.7135	0.6981	0.6517	0.6121
6	0.8203	0.8111	0.8060	0.8008	0.7734	0.7497	0.7306	0.7151	0.6670	0.6243
7	0.8225	0.8159	0.8130	0.8095	0.7867	0.7648	0.7464	0.7310	0.6816	0.6364
8	0.8243	0.8199	0.8188	0.8168	0.7985	0.7785	0.7608	0.7457	0.6956	0.6481
9	0.8258	0.8233	0.8238	0.8232	0.8090	0.7909	0.7741	0.7595	0.7090	0.6595
10	0.8270	0.8262	0.8282	0.8288	0.8184	0.8022	0.7864	0.7723	0.7218	0.6707
20	0.8338	0.8422	0.8525	0.8605	0.8759	0.8748	0.8689	0.8613	0.8222	0.7668
30	0.8365	0.8488	0.8629	0.8743	0.9023	0.9097	0.9103	0.9080	0.8843	0.8374
40	0.8379	0.8524	0.8686	0.8820	0.9170	0.9292	0.9336	0.9345	0.9228	0.8874
50	0.8388	0.8546	0.8723	0.8868	0.9263	0.9413	0.9479	0.9507	0.9469	0.9221
100	0.8407	0.8595	0.8800	0.8972	0.9455	0.9653	0.9749	0.9802	0.9879	0.9865

Table 2.3. Values of $\Delta_w(s_u^2 | s_S^2)$ for various m and all p.

$m =$	2	3	4	5	10	15	20	25	50	100
All p	0.7793	0.7364	0.7102	0.6916	.06414	0.6173	0.6024	0.5921	0.5658	0.5468

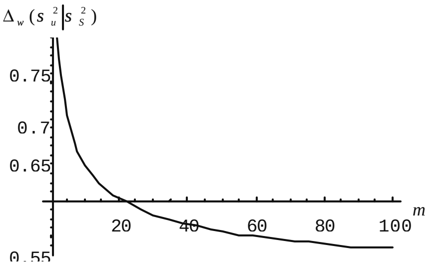

Figure2.3. 3-D graph of $\Delta_w(s_u^2 | s_S^2)$.

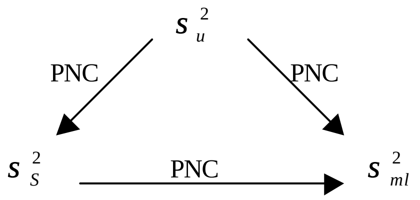

Figure 2.4. Comparison of s_u^2, s_S^2 and s_{ml}^2.

<u>Remark 2.1.</u>

(1) From the above Table 2.1-2.3, it is clear that

(i) s_S^2 is better than s_{ml}^2 (in PNC);

(ii) s_u^2 is better than s_{ml}^2 (in PNC); and

(iii) s_u^2 is better than s_S^2 (in PNC).

The above diagram Figure 2.4 gives a better visual comparison of the three above-mentioned estimators. We thus conclude that s_u^2 (UMVUE as well as the BAEE under L_E) is the best among the three affine equivariant estimators we have discussed above.

(2) In Tables 2.1 and 2.2, for a fixed `$m$', the values of $\Delta_w(s_S^2 | s_{ml}^2)$ and $\Delta_w(s_u^2 | s_{ml}^2)$ are both increasing as $p$ increases. This is due to the fact that in (2.4) and (2.5), both $c_1^*$ and $c_2^*$ are increasing functions of $p$ for a fixed `m'. However, such a simple trend does not occur always if one varies m for a fixed `$p$'. Table 2.1 shows that $\Delta_w(s_S^2 | s_{ml}^2)$ decreases monotonically as $m$ increases ($p$ fixed) in most of the cases ($1 \quad p \quad 50$). Only for `vary large p' ($p = 100$), the values first decrease, then increase and then finally decrease with increasing m. In Table 2.2, values of $\Delta_w(s_u^2 | s_{ml}^2)$ are steadily decreasing as m increases for $1 \quad p \quad 8$. For $p \quad 9$, these values again show a "sine curve trend" (i.e., increase-decrease-increase) as m increases. In Table 2.3, $\Delta_w(s_u^2 | s_S^2)$ decreases monotonically as m increases.

2.2 Asymptotic values of Δ_w 's.

In this subsection we will see the limiting values of Δ_w 's ((2.4) - (2.6)) when (i) p fixed and m ; (ii) $m = p$.

Note that $(c^2_{m-1}/(m-1))$ can be treated as the average of $(m-1)$ *iid* c^2_1 random variates. To be precise, let $Y_1, Y_2, \cdots, Y_{m-1}$ be *iid* c^2_1 -random variables. Then $\overline{Y} = \sum_{i=1}^{m-1} Y_i/(m-1)$ is equivalent to $(c^2_{m-1}/(m-1))$ as far as probability distribution is concerned. Since c^2_1 -distribution has mean 1 and variance 2, by Central Limit Theorem,

$$Y = \sqrt{\frac{m-1}{2}}(\overline{Y}-1) \text{ is asymptotically } N(0,1) . \tag{2.7}$$

We will use (2.7) to study (2.4) - (2.6) asymptotically.

(1) p **fixed and large** m **.**

Note that by using the representation of Y in (2.7),

(i) $\Delta_w(s^2_S \big| s^2_{ml}) = P(c^2_{m-1} \quad c^*_1)$ (from(2.4))

$$= P \quad \sqrt{\frac{m-1}{2}}(\overline{Y}-1) \quad \frac{(mp+4m+3p+4)\sqrt{m-1}}{(2m^2+mp-p-2)\sqrt{2}}$$

$$P(N(0,1) \quad 0) \qquad \text{(for fixed } p \text{ and } m >> p\text{)}$$

(ii) $\Delta_w(s^2_u \big| s^2_{ml}) = P(c^2_{m-1} \quad c^*_2)$ (from (2.5))

$$= P \quad \sqrt{\frac{m-1}{2}}(\overline{Y}-1) \quad \frac{(p+2)\sqrt{m-1}}{(2m+p)\sqrt{2}}$$

$$P(N(0,1) \quad 0) \quad \text{(for fixed } p \text{ and } m >> p\text{)}$$

$$= 0.5$$

Similarly,

(iii) $\Delta_w(s^2_u \big| s^2_S) = P \quad \sqrt{\frac{m-1}{2}}(\overline{Y}-1) \quad \frac{\sqrt{m-1}}{\sqrt{2m}} \quad 0.5$ for large m .

This tells that, in Tables 2.1 - 2.3, each row converges to 0.5 (for any fixed p).

(2) $m = p$ **= large**

Interestingly, we get quite different results when m and p increase simultaneously. This will explain why the diagonal elements $(m = p)$ in Tables 2.1 and 2.2 are converging to 1.

By setting $m = p$, we get

(i) $\Delta_w(s^2_S \big| s^2_{ml}) = P \quad \sqrt{\frac{m-1}{2}}(\overline{Y}-1) \quad \frac{(m^2+m+4)\sqrt{m-1}}{(3m^2-m-2)\sqrt{2}} \qquad 1$ as m ;

and

(ii) $\Delta_w(s^2_u \big| s^2_{ml}) = P \quad \sqrt{\frac{m-1}{2}}(\overline{Y}-1) \quad \frac{(m+2)\sqrt{m-1}}{\sqrt{2}(3m)} \qquad 1$ as m .

3. COMPARISON WITH SCALE EQUIVARIANT SHRINKAGE ESTIMATORS

In Section 2 we found that the unbiased estimator s^2_u, which is also the BAEE under L_E (i.e., s^2_E), is the most preferable in terms of PNC, among the three affine equivariant estimators. But in a decision theoretic setup, under the loss L_E the estimator $s^2_u = s^2_E$ is dominated by the Stein type estimator

$$s^2_{E(S)} = min\left\{S/(m-1), (S+\|X\|^2)/(m+p-1)\right\}, \tag{3.1}$$

and the Brewster-Zidek type estimator

$$s^2_{E(BZ)} = \left\{1 - f_{E(BZ)}(U)\right\} S/(m-1) , \tag{3.2}$$

where $U = S/(S+\|X\|^2)$ and

$$f_{E(BZ)}(U) = \frac{2(m+p-1)^{-1}(1-U)^{p/2}U^{(m-3)/2}}{{}_{U}^{1}(1-y)^{p/2-1}y^{(m-3)/2}dy}. \tag{3.3}$$

The estimator $s^2_{E(S)}$ can be expressed as (similar to $s^2_{E(BZ)}$)

$$s^2_{E(S)} = \{1 - f_{E(S)}(U)\}S/(m-1), \tag{3.4}$$

where

$$f_{E(S)}(U) = max \quad 0, 1 - \frac{(m-1)}{(m+p-1)U} \quad . \tag{3.5}$$

Both the improved estimators in (3.2) and (3.4) have the general structure $s^2_* = s^2_u(1-f(U))$, where $0 < f(U) \quad 1$, and f is nondecreasing in U. So, the probability that $s^2_* = s^2_u(1-f(U))$ is closer to s^2 than s^2_u is so to s^2 is

$$\Delta_w(s^2_* | s^2_u) = P\left[(s^2_* - s^2)^2 \quad (s^2_u - s^2)^2\right]$$

$$= P \;\; f(U)(\frac{S/s^2}{(m-1)}) \quad 2(\frac{S/s^2}{(m-1)} - 1) \quad . \tag{3.6}$$

First we simplify the expression (3.6), and then using specific choices for $f(U)$ (i.e., $f_{E(S)}$, $f_{E(BZ)}$), we get the desired probabilities.

For notational simplicity let $U_1 = S/s^2$ and $U_2 = \|X\|^2/s^2$. Note that U_1 and U_2 follow respectively c^2_{m-1} and $c^2_p(l)$ (noncentral Chi-square distribution with p df and noncentrality parameter $l = \|m\|^2/s^2$), and they are independent. We use the representation: $c^2_p(l) = c^2_{p+2J}$ where J follows Poisson $(l/2)$. Given $J = j$, U_2 follows c^2_{p+2j}; and hence $(U_1 + U_2)$ follows $c^2_{m+p-1+2j}$ which is independent of $U = U_1/(U_1+U_2)$, U follows $Beta((m-1)/2, (p/2)+j)$ distribution. Expression (3.6) can be written as

$$\Delta_w(s^2_* | s^2_u) = P[Uf(U)(U_1+U_2)/(m-1) \quad 2(U(U_1+U_2)/(m-1)-1)]$$

$$= P[(U_1+U_2)U(2-f(U)) \quad 2(m-1)]$$

$$= \sum_{j=0} \exp(-l/2)\frac{(l/2)^j}{j!} P \;\; c^2_{m+p-1+2j} \quad \frac{2(m-1)}{U(2-j(U))} \quad ,$$

(where U is a $Beta((m-1)/2, (p/2)+j)$ random variable)

$$= \sum_{j=0} \exp(-l/2)\frac{(l/2)^j}{j!} \frac{{}_0^1 u^{\frac{m-1}{2}-1}(1-u)^{\frac{p}{2}+j-1}}{Beta(\frac{m-1}{2}, \frac{p}{2}+j)} P \;\; c^2_{m+p-1+2j} \quad \frac{2(m-1)}{u(2-f(u))} \; du \, . \tag{3.7}$$

By using $f(U) = f_{E(BZ)}(U)$ and $f(U) = f_{E(S)}(U)$ in (3.7) we get the expressions for $\Delta_w(s^2_{E(BZ)} | s^2_u)$ and $\Delta_w(s^2_{E(S)} | s^2_u)$ respectively.

In Table 3.1 (a)-(g) and 3.2 (a)-(g) we have computed the values of $\Delta_w(s^2_{E(BZ)} | s^2_u)$ and $\Delta_w(s^2_{E(S)} | s^2_u)$ for $l = 0.0,\ 0.1,\ 0.3,\ 0.5,\ 1.0,\ 5.0,\ 10.0$, and various combinations of m and p. Note that all the tabulated values are less than 0.5.

Table 3.1 (a). Values of $\Delta_w(s^2_{E(BZ)} | s^2_u)$ for $l = 0.0$.

p \ m	5	10	15	20	25	50	100
1	0.3610	0.3938	0.4066	0.4134	0.4174	0.4238	0.4211
2	0.3461	0.3886	0.4076	0.4191	0.4270	0.4467	0.4599
3	0.3278	0.3741	0.3953	0.4083	0.4173	0.4405	0.4575
4	0.3130	0.3620	0.3847	0.3988	0.4086	0.4341	0.4528
5	0.3008	0.3518	0.3757	0.3906	0.4011	0.4284	0.4487
10	0.2616	0.3165	0.3438	0.3612	0.3737	0.4070	0.4327
25	0.2121	0.2661	0.2978	0.3156	0.3322	0.3669	0.4040
50	0.1977	0.2387	0.2662	0.2827	0.2994	0.3389	0.3762

Table 3.1 (b). Values of $\Delta_w(s^2_{E(BZ)} | s^2_u)$ for $l = 0.1$.

p \ m	5	10	15	20	25	50	100
1	0.3619	0.3950	0.4079	0.4148	0.4190	0.4258	0.4238
2	0.3447	0.3893	0.4083	0.4197	0.4276	0.4472	0.4604
3	0.3283	0.3748	0.3960	0.4090	0.4180	0.4411	0.4578
4	0.3135	0.3628	0.3855	0.3995	0.4093	0.4346	0.4532
5	0.3013	0.3525	0.3765	0.3914	0.4019	0.4290	0.4491
10	0.2620	0.3173	0.3445	0.3620	0.3744	0.4077	0.4332
25	0.2063	0.2548	0.2832	0.3016	0.3122	0.3523	0.3855
50	0.1858	0.2296	0.2521	0.2689	0.2838	0.3217	0.3565

Table 3.1 (c). Values of $\Delta_w(s^2_{E(BZ)} | s^2_u)$ for $l = 0.3$.

p \ m	5	10	15	20	25	50	100
1	0.3634	0.3972	0.4104	0.4175	0.4219	0.4295	0.4286
2	0.3475	0.3906	0.4096	0.4210	0.4288	0.4482	0.4613
3	0.3292	0.3762	0.3974	0.4103	0.4193	0.4421	0.4586
4	0.3144	0.3642	0.3870	0.4010	0.4107	0.4357	0.4541
5	0.3022	0.3540	0.3781	0.3929	0.4033	0.4301	0.4500
10	0.2627	0.3187	0.3461	0.3635	0.3760	0.4090	0.4342
25	0.1873	0.2333	0.2539	0.2699	0.2868	0.3198	0.3460
50	0.1680	0.2057	0.2295	0.2422	0.2563	0.2941	0.3208

Table 3.1 (d). Values of $\Delta_w(s^2_{E(BZ)} | s^2_u)$ for $l = 0.5$.

p \ m	5	10	15	20	25	50	100
1	0.3648	0.3993	0.4128	0.4201	0.4246	0.4329	0.4331
2	0.3484	0.3918	0.4108	0.4221	0.4299	0.4492	0.4622
3	0.3302	0.3776	0.3988	0.4116	0.4205	0.4430	0.4593
4	0.3154	0.3656	0.3885	0.4024	0.4121	0.4368	0.4549
5	0.3031	0.3555	0.3796	0.3944	0.4047	0.4313	0.4509
10	0.2634	0.3200	0.3476	0.3651	0.3775	0.41032	0.4352
25	0.1686	0.2074	0.2313	0.2454	0.2579	0.2874	0.3142
50	0.1509	0.1880	0.2048	0.2194	0.23203	0.2638	0.2935

Table 3.1 (e). Values of $\Delta_w(s^2_{E(BZ)} | s^2_u)$ for $l = 1.0$.

p \ m	5	10	15	20	25	50	100
1	0.3681	0.4038	0.4178	0.4255	0.4304	0.4401	0.4425
2	0.3505	0.3948	0.4137	0.4249	0.4325	0.4514	0.4642
3	0.3324	0.3809	0.4020	0.4147	0.4234	0.4454	0.4611
4	0.3176	0.3692	0.3920	0.4058	0.4153	0.4394	0.4568
5	0.3053	0.3591	0.3833	0.3979	0.4081	0.4340	0.4529
10	0.2653	0.3234	0.3513	0.3688	0.3812	0.4134	0.4377
25	0.1325	0.1629	0.1808	0.1922	0.1983	0.2229	0.2435
50	0.1182	0.1459	0.1632	0.1733	0.1792	0.2056	0.2263

Table 3.1 (f). Values of $\Delta_w(s^2_{E(BZ)} | s^2_u)$ for $l = 5.0$.

p \ m	5	10	15	20	25	50	100
1	0.36823	0.4227	0.4379	0.4463	0.4519	0.4649	0.4733
2	0.3636	0.4120	0.4301	0.44025	0.4469	0.4629	0.4737
3	0.3471	0.4012	0.4217	0.4331	0.4407	0.4587	0.4710
4	0.3329	0.3914	0.4139	0.4265	0.4348	0.4546	0.4681
5	0.3207	0.3825	0.4067	0.4203	0.4293	0.4508	0.4654
10	0.2788	0.3474	0.3771	0.3943	0.4059	0.4341	0.4536
25	0.0177	0.0222	0.0244	0.0260	0.0273	0.0305	0.0332
50	0.0162	0.0198	0.0220	0.0235	0.0247	0.0278	0.0309

Table 3.1 (g). Values of $\Delta_w(s^2_{E(BZ)} | s^2_u)$ **for** $l = 10.0$ **.**

p \ m	5	10	15	20	25	50	100
1	0.3842	0.4247	0.4390	0.4771	0.4522	0.4645	0.4729
2	0.3688	0.4176	0.4343	0.4432	0.4490	0.4627	0.4718
3	0.3545	0.4103	0.4291	0.4392	0.4456	0.4605	0.7404
4	0.3417	0.4030	0.4239	0.4350	0.4421	0.4583	0.4689
5	0.3303	0.3959	0.4188	0.4308	0.43855	0.4560	0.4674
10	0.2886	0.3649	0.3946	0.4107	0.4211	0.4447	0.4597
25	0.0015	0.0018	0.0020	0.0021	0.0022	0.0025	0.0027
50	0.0013	0.0016	0.0018	0.0019	0.0020	0.0023	0.0025

Table 3.2 (a). Values of $\Delta_w(s^2_{E(S)} | s^2_u)$ **for** $l = 0.0$ **.**

p \ m	5	10	15	20	25	50	100
1	0.3690	0.4023	0.4145	0.4207	0.4243	0.4291	0.4250
2	0.3689	0.4075	0.4243	0.4342	0.4409	0.4571	0.4675
3	0.3645	0.4025	0.4197	0.4301	0.4372	0.4553	0.4681
4	0.3620	0.3988	0.4159	0.4264	0.4338	0.4525	0.4662
5	0.3605	0.3961	0.4130	0.4235	0.4310	0.4502	0.4644
10	0.3595	0.3897	0.4048	0.4148	0.4220	0.4420	0.4578
25	0.3652	0.3885	0.3993	0.4068	0.4126	0.4304	0.4468
50	0.3724	0.3932	0.4012	0.4064	0.4103	0.4234	0.4380

Table 3.2 (b). Values of $\Delta_w(s^2_{E(S)} | s^2_u)$ **for** $l = 0.1$ **.**

p \ m	5	10	15	20	25	50	100
1	0.3706	0.4038	0.4161	0.4223	0.4260	0.4311	0.4277
2	0.3702	0.4086	0.4252	0.4350	0.4416	0.4577	0.4680
3	0.3658	0.4034	0.4206	0.4309	0.4380	0.4558	0.4686
4	0.3632	0.3999	0.4169	0.4273	0.4346	0.4531	0.4666
5	0.3618	0.3972	0.4140	0.4245	0.4318	0.4508	0.4649
10	0.3605	0.3907	0.4058	0.4157	0.4229	0.4427	0.4583
25	0.3658	0.3892	0.4001	0.4075	0.4133	0.4310	0.4474
50	0.3728	0.3936	0.4017	0.4069	0.4108	0.4240	0.4385

Table 3.2 (c). Values of $\Delta_w(s^2_{E(S)} | s^2_u)$ **for** $l = 0.3$ **.**

p \ m	5	10	15	20	25	50	100
1	0.3734	0.4067	0.4190	0.4254	0.4291	0.4350	0.4326
2	0.3726	0.4106	0.4269	0.4365	0.4430	0.4588	0.4690
3	0.3683	0.4057	0.4224	0.4325	0.4395	0.4569	0.4693
4	0.3657	0.4021	0.4188	0.4290	0.4362	0.4543	0.4675
5	0.3641	0.3994	0.4159	0.4262	0.4334	0.4520	0.4657
10	0.3624	0.3927	0.4046	0.4174	0.4245	0.4440	0.4593
25	0.3670	0.3906	0.4016	0.4090	0.4147	0.4322	0.4484
50	0.3735	0.3946	0.4028	0.4080	0.4119	0.4251	0.4394

Table 3.2 (d). Values of $\Delta_w(s^2_{E(S)} | s^2_u)$ **for** $l = 0.5$ **.**

p \ m	5	10	15	20	25	50	100
1	0.3762	0.4093	0.4217	0.4282	0.4320	0.4385	0.4371
2	0.3750	0.4124	0.4285	0.4380	0.4443	0.4598	0.4699
3	0.3706	0.4077	0.4242	0.4341	0.4409	0.4579	0.4701
4	0.3680	0.4041	0.4206	0.4307	0.4376	0.4554	0.4683
5	0.3664	0.4014	0.4178	0.4279	0.4350	0.4532	0.4666
10	0.3643	0.3946	0.4094	0.4191	0.4261	0.4452	0.4602
25	0.3681	0.3920	0.4030	0.4104	0.4161	0.4335	0.4494
50	0.3742	0.3955	0.4038	0.4090	0.4130	0.4261	0.4403

Table 3.2 (e). Values of $\Delta_w(s^2_{E(S)} | s^2_u)$ for $l = 1.0$.

p \ m	5	10	15	20	25	50	100
1	0.3820	0.4149	0.4274	0.4340	0.4382	0.4458	0.4466
2	0.3800	0.4166	0.4320	0.4411	0.4472	0.4621	0.4719
3	0.3759	0.4121	0.4280	0.4375	0.4440	0.4602	0.4718
4	0.3733	0.4087	0.4247	0.4343	0.4410	0.4579	0.4701
5	0.3716	0.4061	0.4219	0.4317	0.4385	0.4558	0.4685
10	0.3686	0.3990	0.4136	0.4231	0.4299	0.4482	0.4624
25	0.3709	0.3953	0.4064	0.4138	0.4195	0.4364	0.4517
50	0.3760	0.3978	0.4063	0.4116	0.4157	0.4287	0.4426

Table 3.2 (f). Values of $\Delta_w(s^2_{E(S)} | s^2_u)$ for $l = 5.0$.

p \ m	5	10	15	20	25	50	100
1	0.4017	0.4335	0.4460	0.4532	0.4579	0.4691	0.4761
2	0.4000	0.4325	0.4457	0.4532	0.4583	0.4707	0.4792
3	0.3981	0.4307	0.4440	0.4518	0.4570	0.4698	0.4787
4	0.3965	0.4291	0.4425	0.4504	0.4557	0.4687	0.4779
5	0.3952	0.4276	0.4411	0.4490	0.4544	0.4678	0.4772
10	0.3912	0.4222	0.4355	0.4435	0.4491	0.4633	0.4738
25	0.3876	0.4153	0.4270	0.4343	0.4396	0.4540	0.4658
50	0.3873	0.4127	0.4227	0.4287	0.4331	0.4456	0.4572

Table 3.2 (g). Values of $\Delta_w(s^2_{E(S)} | s^2_u)$ for $l = 10.0$.

p \ m	5	10	15	20	25	50	100
1	0.4000	0.4309	0.4432	0.4502	0.4549	0.4663	0.4741
2	0.3996	0.4306	0.4430	0.4501	0.4548	0.4663	0.4742
3	0.3991	0.4302	0.4426	0.4497	0.4545	0.4663	0.4741
4	0.3986	0.4297	0.4422	0.4493	0.4541	0.4658	0.4739
5	0.3982	0.4292	0.4417	0.4489	0.4538	0.4655	0.4737
10	0.3963	0.4270	0.4386	0.4469	0.4518	0.4639	0.4725
25	0.3929	0.4223	0.4343	0.4414	0.4463	0.4589	0.4683
50	0.3908	0.4186	0.4296	0.4361	0.4406	0.4525	0.4623

Concluding Remark: Even though the scale equivariant (but not location in variant) estimators $s^2_{E(BZ)}$ and $s^2_{E(S)}$ are better than s^2_u (UMVUE) in terms of risk, the latter is much superior to formers in terms of PNC, and therefore, apart from being affine equivariant, the UMVUE seems more appealing as an estimator of s^2.

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