MULTIPLE ATTRIBUTE DOMINANCE RULES WITH WEIGHT AND VALUE IMPRECISION

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Abstract

Multiple attribute decision methods with imprecision on decision parameters such as attribute weights and performance scores are presented. Such imprecision, on one hand, may give decision-maker chances that are enhanced freedom of choice and comforts of specification and, on the other hand, may cause decision analysts difficulty of establishing dominance relationships among alternatives. To establish dominance relationships, we revisit three dominance rules such as strict dominance rule, pairwise strict dominance rule, and pairwise weak dominance rule under imprecision on both weight and value, and investigate their properties, which have been done under at best either weight or value imprecision.

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1. Introduction

Decision making involves choosing some course of action among various alternatives. In almost all decision making problems, there are several conflicting criteria for judging alternatives. A multiple attribute decision making (MADM) method usually consists of two phases: 1) construction and information input, and 2) aggregation and exploitation. In the first phase, the decision-maker is asked to determine the alternative and criteria and provide the performance information of alternatives with respect to each criterion and possibly the relative importance (i.e., weights) of criteria. In the second phase, we select alternative with the highest value score which alternative can attain. Related with parameter input of the first phase, the types of preference information allowed in research models differ from exact parameter estimates to imprecise data. Earlier studies on MADM with imprecise preference information can be found in the literature such as in [1-10]. Instead of the term 'imprecise information', some authors refer it to 'incomplete information,' 'partial information,' 'linear partial information (LPI) or 'incomplete knowledge,' but all of these are a little different to each other and can be included by linear inequalities of preferences such as rankings and bounded descriptions.

In the second phase, Sage and White [8] have spearheaded an aggregation method of imprecise preference information about *both weight and utility* by proposing ISMAUT (imprecisely specified multiattribute utility theory). The method is an extended version in that it explicitly allows both weight and utility imprecision although they are in restricted forms and, so far, only a few studies have employed imprecision on weight and utility information. Compared with prior works, we extend imprecision on attribute weight and performance score to cover more general preferential expressions, and dominance rules available for ranking or screening alternatives under imprecise preference data are revisited and properties are exploited.

2. Preliminaries and definitions

In multiple attribute decision making (MADM), one usually considers a finite discrete set of alternatives, $A = \{x_1, x_2, \ldots, x_M\}$, which is valued by a finite discrete set of attributes, $K = \{1, 2, \ldots, N\}$. Let $v_k(x_i)$ be the value of alternative $x_i \in A$ on attribute $k \in K$ and w_k a scaling factor to represent the relative importance of the kth attribute. A classical evaluation of alternative leads to the aggregation of all criteria into a unique criterion called value function under certainty and utility function under uncertainty. In this paper, we assume that there exist additive value functions under preferential independence [11] and thus aggregated value function $V(x_i)$ of alternative x_i is denoted as follows:

$$V(x_i) = \sum_{k=1}^{N} w_k v_k(x_i) .$$

2.1 Generation of imprecise preference data

We do not force decision-maker to specify parameters as input data to the extent that this becomes overly stressful or behaviorally and physically irrelevant in view of the inherent imprecision associated with domain knowledge of parameters characterizing the decision situation. To this end, when decision-maker provides his knowledge or preference information on the values $\{w_i\}$ and $\{v_i(\cdot)\}$, it is assumed that the precise values of weights are not known and hence the only information is to satisfy linear constraints such as the forms of: a1) weak ranking, $w_k \ge w_l$, a2) strict ranking, $w_k - w_l \ge \varepsilon$, where ε is a small positive number, a3) preference with multiples, $w_k \ge \alpha_{kl} w_l$, a4) interval preference with numerical ranges, $l_k \le w_k \le u_k$, or a5) preference differences, $w_k - w_l \ge w_m - w_n$, for $k \ne l \ne m \ne n$. For example, a possible weight set W can be $\{(w_k, w_l) \mid w_k \ge 2 \cdot w_l, 0.2 \le w_k \le 0.5, w_k + w_l = 1, w_k, w_l \ge 0\}$.

It is also assumed that the values $\{v_i(\cdot)\}$ are not known exactly and specified in the form of arbitrary linear equalities such as b1) weak preference, $v_k(x_i) \ge v_k(x_j)$, b2) strict preference, $v_k(x_i) - v_k(x_j) \ge \varepsilon$, where ε is a small positive number, b3) preference with multiples, $v_k(x_i) \ge \alpha_{ij} \cdot v_k(x_j)$, b4) interval preference, $l_i \le v_k(x_i) \le u_i$, or b5) preference differences, $v_k(x_i) - v_k(x_j) \ge v_k(x_l) - v_k(x_m)$, for $i \ne j \ne l \ne m$. Let V_k be the set of constraints on the values $v_k(\cdot)$, considering the outcomes of alternatives for each attribute index, $k \in K$. Without loss of generality, we assume that the set of V_k for each $k \in K$ can include arbitrary linear constraints as well as non-negativity sign restrictions on the variables $v_k(\cdot)$, for example, $V_k = \{v_k(x_i), v_k(x_j) \mid v_k(x_i) \ge 3 \cdot v_k(x_j), 0.6 \le v_k(x_i) \le 0.8\}$. Let Ω be set of collecting dominance relations between the alternatives, $\Omega \subseteq A \times A$, for example, $(x_i, x_j) \in \Omega$ means that alternative x_i is at least as preferred as alternative x_j.

2.2 Definition of dominance rules

Definition 1: SD (strict dominance) relationship holds between alternative x_i and x_j if and only if it holds that $\varsigma_{\min}(x_i) \ge \varsigma_{\max}(x_j)$, where

$$\varsigma_{\min(\max)}(x_i) = \min\ (\max)\ \{\textstyle\sum_{k \in K} w_k v_k(x_i) \mid w_k \in W, v_k(x_i) \in V_k, \forall k \in K\}. \tag{1}$$

Definition 2: PSD (pairwise strict dominance) relationship holds between alternative x_i and x_j if and only if it holds that $\varsigma_{\min}(x_i, x_j) \ge 0$, where

$$\varsigma_{\min}(x_i, x_j) = \min\ \{\textstyle\sum_{k \in K} w_k [v_k(x_i) - v_k(x_j)] \mid w_k \in W, v_k(\cdot) \in V_k, \forall k \in K\}. \tag{2}$$

Definition 3: PWD (pairwise weak dominance) relationship holds between alternative x_i and x_j if and only if it holds that $\varsigma_{\min}(x_i, x_j) \ge \varsigma_{\min}(x_j, x_i)$ (note that the order is reversed).

3. Some Theories on Dominance Rules with Imprecise Data

As can be seen, dominance rules in Section 2 become nonlinear programming problems with product form of weight and value which range $w_k \in W$ and $v_k \in V_k$, $k \in K$, respectively and hence can not be treated by standard methods without further alternation like those we present below. We can employ the techniques developed by Sage and White (1984) and the nonlinear program is separable into two linear programs because each decision variable takes values in its independent decision space. Mathematical programming for identifying PSD relationship in Definition 2, for example, can be reduced to a series of linear programs.

$$\varsigma_{\min}(x_i, x_j) = \min\ \{\textstyle\sum_{k \in K} w_k \delta_k(x_i, x_j) \mid w_k \in W\},$$

where
$\delta_k(x_i, x_j) = \min\{\sum k \in K\ [v_k(x_i) - v_k(x_j)] \mid v_k(\cdot) \in V_k,\ k \in K\}$.

To construct Ω_{SD} and Ω_{PSD} (or Ω_{PWD}), it is needed to solve at most $2M(N+1)$ and $M(M-1)(N+1)$ linear programs, respectively.

Theorem 1: Let Ω_{SD} be the set of preference orders determined by SD rule, Ω_{PSD} by PSD rule, and Ω_{PWD} by PWD rule. According to three dominance rules, following set relationships hold:

$$\Omega_{SD} \subseteq \Omega_{PSD} \subseteq \Omega_{PWD}.$$

Proof.

(i) Let us prove the first assertion $\Omega_{SD} \subseteq \Omega_{PSD}$, which implies that if there exists SD relationship among alternatives, PSD relationship always holds among those alternatives. Suppose that SD relationship between alternative x_i and x_j holds, i.e., $\varsigma_{\min}(x_i) \geq \varsigma_{\max}(x_j)$. It implies that $\varsigma_{\min}(x_i) - \varsigma_{\max}(x_j) = \varsigma_{\min}(x_i) - \{-\varsigma_{\max}(-x_j)\} \geq 0$, where $\varsigma_{\max}(-x_j) = \min\{\sum_{k \in K} -w_k v_k(x_j) \mid w_k \in W, v_k(x_j) \in V_k\}$, and, in turn, becomes $\varsigma_{\min}(x_i) + \varsigma_{\min}(-x_j) \geq 0$. It is derived that $\varsigma_{\min}(x_i, x_j) \geq \varsigma_{\min}(x_i) + \varsigma_{\min}(-x_j) \geq 0$ because optimized value of sum of weighted differences under common weight set, i.e., $\varsigma_{\min}(x_i, x_j)$ is always greater than or equal to sum of individually optimized values under individual weight set, i.e., $\varsigma_{\min}(x_i) + \tau_{\min}(-x_j)$.

(ii) Let us prove the second assertion $\Omega_{PSD} \subseteq \Omega_{PWD}$, which implies that if there exists PSD relationship among alternatives, PWD relationship holds among those alternatives. Let us suppose that PSD relationship between alternative x_i and x_j hold, i.e., $\varsigma_{\min}(x_i, x_j) \geq 0$ and rewrite it as $\varsigma_{\min}(x_i, x_j) = \varsigma_{\min}\text{-}(x_j, x_i)$, where
$\varsigma_{\min}\text{-}(x_j, x_i) = \min\{\sum_{k \in K} -w_k[v_k(x_j) - v_k(x_i)] \mid w_k \in W, v_k(x_i), v_k(x_j) \in V_k\}$. It holds that
$\varsigma_{\min}\text{-}(x_j, x_i) = -\varsigma_{\max}(x_j, x_i) \geq 0$ and, in turn, $\varsigma_{\min}(x_j, x_i) \leq \varsigma_{\max}(x_j, x_i) \leq 0$, which implies $\varsigma_{\min}(x_i, x_j) \geq \varsigma_{\min}(x_j, x_i)$.

As can be seen in Definition 3, it is always possible to establish dominance relationships among alternatives by PWD rule. Thus, Ω_{PWD} is nonempty and has elements of $M!/\{(M-2)!\cdot 2!\}$. The number of identified preference orders increases from SD rule to PWD rule according to Theorem 1. However, rules for establishing dominance with imprecise data are confined to use SD or PSD rule in spite of increase of identified preference orders by PWD rule. Although the number of linear programs for establishing dominance relationship is the least with SD rule, it rarely produces rank order among alternatives because it requires so *strict* dominance constraint. Therefore, PSD rule is used under imprecise preference data.

Theorem 2: SD and PSD rules are transitive, but PWD rule is not always transitive.

Proof.

(i) SD rule has transitive property if $\varsigma_{\min}(x_i) \geq \varsigma_{\max}(x_j)$ and $\varsigma_{\min}(x_j) \geq \varsigma_{\max}(x_l)$, then $\varsigma_{\min}(x_i) \geq \varsigma_{\max}(x_l)$. This statement can be easily proved due to the fact of $\varsigma_{\max}(x_j) \geq \varsigma_{\min}(x_j)$ and thus $\varsigma_{\min}(x_i) \geq \varsigma_{\max}(x_j) \geq \varsigma_{\min}(x_j) \geq \varsigma_{\max}(x_l)$.
PSD rule has transitive property if $\varsigma_{\min}(x_i, x_j) \geq 0$ and $\varsigma_{\min}(x_j, x_l) \geq 0$, then $\varsigma_{\min}(x_i, x_l) \geq 0$. If we rewrite the conclusion part and use proof (i) part in Theorem 1, $0 \leq \varsigma_{\min}(x_i, x_j) + \varsigma_{\min}(x_j, x_l) \leq \varsigma_{\min}(x_i, x_j, x_l) = \varsigma_{\min}(x_i, x_l)$,
where, $\varsigma_{\min}(x_i, x_j, x_l) = \min\{\sum_{k \in K} w_k\{[v_k(x_i) - v_k(x_j)] + [v_k(x_j) - v_k(x_l)]\}\}$.

(ii) PWD rule is not always transitive. If $\Omega_{PWD} = \Omega_{PSD}$, then PWD rule is transitive but if $\Omega_{PSD} \subset \Omega_{PWD}$, then PWD rule is not always transitive. The intransitivity of PWD rule (more exactly intransitivity between elements in $\Omega_{PWD} - \Omega_{PSD}$)

can be proved by a counterexample. Suppose that the performance scores are given in Table 1 and weight information is given such as $W = \{(w_1, w_2) | w_1 \geq 3 \cdot w_2, w_1 + w_2 = 1, w_1, w_2 \geq 0\}$.

Table 1 An artificial example with two attributes and three alternatives

Alternative	Attribute 1	Attribute 2
x	[0.6, 0.8]	[0.3, 0.49]
y	[0.59, 0.7]	[0.4, 0.8]
z	[0.62, 0.77]	[0.31, 0.52]

The extreme points of weight space are {(0.75, 0.25), (1,0)}. The pairwise dominance result for alternative x and y becomes $\varsigma_{\min}(x, y) = \min\{.75 \times (.6-.7) + .25 \times (.3-.8), 1 \times (.6-.7) + 0 \times (.3-.8)\} = -0.2$. Similarly, we can obtain $\zeta_{\min}(y, x) = -0.21$, $\zeta_{\min}(y, z) = -0.18$ $\zeta_{\min}(z, y) = -0.1825$, $\zeta_{\min}(x, z) = -0.1825$, and $\zeta_{\min}(z, x) = -0.18$. This implies that $\Omega_{PWD} = \{(x, y), (y, z), (z, x)\}$ is intransitive.

Remark 1: PWD relationship of alternative x_i over x_j does ensure that the weight and value score most favorable to alternative x_i yield a higher expected difference than does the weight and value score most favorable to alternative x_j. If we choose a pairwise weakly dominant alternative when there are two competing non-dominated alternatives, it means that we are trying to select an alternative with less "regret" as traditionally defined [5, 7].

4. Numerical illustration

This section illustrates features of various dominance rules in the context of a modified example [12]. The aim of the example is to help graduating students evaluate alternative employment options. There are three employment alternatives: a government office (x); a large enterprise with a good reputation (y); small company financed by venture capital (z). In the original example, there are two-leveled hierarchical attributes classified under the attributes job security, income, and career opportunities, but in this paper, we consider stability of the firm (w_S), annual starting salary (w_A), future salary increases (w_F), educational opportunities (w_E). The performance information on four criteria is shown in Table 2. This information is artificially made for emphasizing the dominance rules under imprecision on weight and value scores.

Table 2 Imprecise data on performance

Attribute	Imprecise data on performance
Stability of the firm	$v_S(x) \geq v_S(y), v_S(y) \geq 3 \cdot v_S(z), 0.6 \leq v_S(x) \leq 0.7$
Annual starting salary	$v_A(z) \geq v_A(y) \geq v_A(x), 0.7 \leq v_A(y) \leq 0.8$ $0.1 \leq v_A(x) \leq 0.2$
Future salary increase	$v_F(z) \geq 1.2 \cdot v_F(y), v_F(y) \geq 2 \cdot v_F(x), v_F(y) + v_F(z) \geq 0.8$
Educational opportunity	$v_E(y) \geq 1.5 \cdot v_E(z), v_E(z) \geq 5 v_E(x), v_E(y) + v_E(z) \geq 0.8$

For example, imprecisely specified value information such as $v_S(x) \geq 3 \cdot v_S(y)$ means that decision-maker feels that a government office is three times as stable as large enterprise. This kind of ratio evaluation can be widely found in AHP literature (Saaty, 1980). Furthermore, suppose that the decision-maker makes preference statements between attributes: 1) annual starting salary is more important than the sum of stability of the firm and educational opportunity, and future salary increase is more important than annual starting salary 2) stability of the firm is twice as important as educational opportunity 3) importance of educational opportunity is between 0.1 and 0.2.

These statements can be summarized such as

$$\Phi_W = \{w_A \geq w_S + w_E,\, w_F \geq w_A,\, w_S \geq 2 \cdot w_E,\, 0.1 \leq w_E \leq 0.2\}.$$

By SD rule, we can obtain interval number for each alternative such as [0.150, 0.382] for alternative *x*, [0.258, 0.813] for *y*, and [0.475, 0.813] for *z*, where lower bound is $\zeta_{\min}(\cdot)$ value and upper bound is $\zeta_{\max}(\cdot)$ respectively in interval number. This implies that $\Omega_{SD} = \varnothing$. By PSD and PWD rule, we can calculate pairwise dominance results such as $\Omega_{PSD} = \{(y, x), (z, x)\}$ and $\Omega_{PWD} = \{(y, x), (z, x), (z, y)\}$ and the computational details are in Table 3.

Table 3 PSD and PWD result

	x	y	z
x	–	-0.643*	-0.657
y	0.045	–	-0.474
z	0.088	-0.176	–

*: $\varsigma_{\min}(x, y)$

The pairwise dominance relationship between paired alternatives is depicted in Figure 1, where solid line indicates PSD and dotted line PWD relationship.

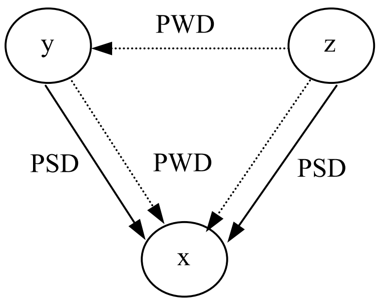

Fig. 1 Graphical presentation of dominance relationship

For the best alternative to choose, there exists unknown dominance relationship based on PSD, i.e., $\Omega_{PWD} - \Omega_{PSD} = \{(z, y)\}$. It depends on the decision situation whether further interactive question and answer with decision-maker are made for the choice of best alternative or whether decision process is stopped to the extent of minimized regrets. For example, if decision-maker considers alternative *z* (i.e., venture company) will make much money in near future and his/her preference changes from $v_F(z) \geq 1.2 \cdot v_F(y)$ into $v_F(z) \geq 7 \cdot v_F(y)$ on future salary increase, pairwise dominance relationship will be changed and computational details are in Table 4.

Table 4 PSD and PWD result with preference changed

	x	y	z
x	–	-0.395	-0.657
y	0.045	–	-0.474
z	0.220	0	–

From this computation, we can construct full PSD results such as $\Omega_{PSD} = \Omega_{PWD} = \{(y, x), (z, x), (z, y)\}$ and conclude that the best alternative to select is alternative *z* for the decision-maker.

5. Concluding remarks

Multiple attribute decision support is conceived to prioritize alternatives under the conflicting alternatives. Sometimes, decision-maker faces the situation where he/she can not express preferences in an exact way because some attributes are intangible or he/she has limited cognitive constraints about the problem domain. To deal with this situation, we suggest methods with imprecise information which takes any forms of linear inequalities such as a1)-a5) and b1)-b5) in Section 2. Meanwhile, three dominance rules for establishing dominance relationship are revisited and their properties are exploited. Mathematical programming for identifying dominance is needed due to the fact that weight and performance scores take any values in imprecisely specified decision space.

However, broadening types of preference information necessitates the deterioration of clear selection of best alternative as is usual case with imprecise data. Therefore, *interactive* features are necessary to restrict the specification of decision-maker's preference and thus, decision support system for systematic consideration of interactive procedure with decision-maker is left for a further research.

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