Selecting An Optimal Hierarchy Structure of AHP Model: An Application of Global Grey Relational Analysis

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Abstract

This paper is develops a novel algorithm that determines selects an optimal hierarchy structure for AHP model. The procedure is based on the global grey relational analysis of grey system, which performs is selection and ranking. The algorithm use sequences to describe hierarchy structure in set from. It then constructs the grey relational matrix and, then, calculates grey relational grades. After doing so, it obtains the optimal hierarchy. Results in this study can be used to select an optimal hierarchy from multiple hierarchical structures for the AHP model.

Keywords: AHP, hierarchy structure, global grey relational analysis.

1. Introduction

This study develops a novel algorithm to determine an optimal hierarchical structure of an analytic hierarchy process (AHP). In 1971, Thomas Saaty developed the AHP process that focuses on the choosing phase of decision-making [5,6]. The AHP provided a hierarchical framework within which problems can be structured. Its use of a ratio scale and paired relative comparisons enable AHP to compare attributes, which performs is selection and ranking [6].

The AHP model provides a methodology to determine the weights of criteria for hierarchical structure. Selecting an optimal hierarchy structure is a tricky problem for the AHP model. For example, Figs. 1 to 5 show hierarchical structure schemas for the teacher research performance [3]. What kind of hierarchical structure yields the optimal estimate? The traditional method cannot solve the optimal hierarchical structure problem, but global grey relational analysis, from grey system theory, can solve the optimal hierarchical structure problem.

Professor Ju-long Deng first developed grey system theory in 1982. Grey system theory provides relational analysis, modeling, prediction and control of grey system, whose structures are latent and ntricate, or whose system information is unclear and deficient [2]. Global grey relational analysis uses global grey relational grade to measure the distance and describes the relationship between sequences of data [1].

This study assumes multiple hierarchical structures for AHP, and a grey relation between the hierarchy structures. Using sequences to describe hierarchy structure and global grey relational analysis can be use to select the optimal hierarchical structure.

2. Proposed Procedure

A novel algorithm is developed to determine the optimal hierarchical structure. The algorithm includes the following four steps - (1) using set form to describe hierarchy structure; (2) using sequences to describe the hierarchy structure; (3) constructing grey relational matrix, and (4) calculating global grey relational grades.

First Step. Use the set form to describe hierarchy structure.

In this step, the weights of criteria are specified as finite ordered sets. Consider some decision problem or proposition, **P**, which has n hierarchical structure schemas. Saaty [7] demonstrated that the i-th hierarchy structure, ${}_iH$, can be written as a finite ordered set,

$$
\begin{aligned}
{}_iH = \Big\{ {}_iC_k^j \mid {}_iC_k^j & \text{ is the } k\text{-th criterion of the } j\text{-th level of the } i\text{-th hierarchy} \\
& \text{and the nature of the } k\text{-th criterion is about criterion of the } j\text{ - 1 level, for each } i, j, h \in \mathbf{N} \Big\},
\end{aligned}
$$

Second Step. Use sequences to describe the hierarchy structure in set form

This step use sequence, ${}_ix$, to describe a hierarchy structure ${}_iH$, and all sequences must to have equal length, as required by global grey relational analysis. Constructing the macro hierarchical structure, TH, can solve the problem that all sequences must to have equal length. The macro hierarchical structure TH is synthesized by all hierarchy structures ${}_iH$, $0 \le i \le l$, and TH is defined as,

$$
\begin{aligned}
{}^TH &= {}_1H \cup {}_2H \cup \cdots \cup {}_lH \\
&= \left\{ {}_1C_k^j \right\} \cup \left\{ {}_2C_k^j \right\} \cup \cdots \cup \left\{ {}_lC_k^j \right\} \\
&= \Big\{ {}^TC_k^j \mid {}^TC_k^j \text{ is the } k\text{-th criterion of the } j\text{-th level of the macro hierarchical structure.} \\
&\qquad , {}^TC_k^j \in {}_1C_k^j \vee {}^TC_k^j \in {}_2C_k^j \vee \cdots \vee {}^TC_k^j \in {}_lC_k^j, \text{ for each } \ j, k \in \mathbf{N} \Big\}.
\end{aligned}
$$

Next, the hierarchy structure ${}_iH$ is indicated the macro hierarchical structure TH, and noted ${}_i^{\mathbf{T}}H$,

$$
\begin{aligned}
{}_i^TH' = \Big\{ {}_i^TC_k^j \mid {}_i^TC_k^j & \text{ is the } k\text{-th criterion of the } j\text{-th level of the macro hierarchical structure.} \\
& , {}_i^TC_k^j \in {}_1C_k^j \vee {}_i^TC_k^j \in {}_2C_k^j \vee \cdots \vee {}_i^TC_k^j \in {}_lC_k^j, \text{ for each } \ j, k \in \mathbf{N} \Big\}.
\end{aligned}
$$

Furthermore, the weight ${}_iw_k^j$ is a priority of criterion ${}_iC_k^j$ of the hierarchy structure ${}_iH$; the weight ${}_i^Tw_k^j$ is a priority of criterion ${}_i^TC_k^j$ of the macro hierarchical structure. For the hierarchy structure ${}_i^{\mathbf{T}}H$ can be quantified that using the priority of criterion ${}_iw_k^j$, represent ${}_iC_k^j$; the macro hierarchy structure ${}_i^{\mathbf{T}}H$ can be quantified that using the priority of criterion, ${}_i^Tw_k^j$ represent, represent ${}_i^TC_k^j$. When the hierarchy structure ${}_iH$ is expressed the macro hierarchical structure ${}_i^{\mathbf{T}}H$, if the criterion ${}_iC_k^j$ have equal to criterion ${}_i^TC_k^j$, then the weight ${}_iw_k^j$ is the weight ${}_i^Tw_k^j$, else the weight ${}_i^Tw_k^j$ is zero. Finally, the macro hierarchy structure ${}_i^{\mathbf{T}}H$ is assumed to be equal as ${}_i^{\mathbf{T}}H'$,

$$
\begin{aligned}
{}_i^TH' = \Big\{ {}_i^Tw_k^j \mid {}_i^Tw_k^j & \text{ is the weight of the } k\text{-th criterion of the } j\text{-th level of the macro hierarchical structure.} \\
& , {}_i^Tw_k^j \in {}_iw_k^j \vee {}_i^Tw_k^j = 0, \text{ for each } \ j, k \in \mathbf{N} \Big\}.
\end{aligned}
$$

This step use sequence ${}_ix$ to describe the macro hierarchy structure ${}_i^{\mathbf{T}}H'$. However, ${}_i^{\mathbf{T}}H'$ cannot be immediately described by sequence ${}_ix$, because the levels of the i-th hierarchy have different degrees of importance. Therefore, a transfer function is need, using which importance of the elements in the sequence ${}_ix$ can be described for the various levels of ${}_i^{\mathbf{T}}H$. The transfer function is assumed to be as follows.

$$_{i}x\left(r\right)=\lambda^{j-1}\,{}_{i}^{T}w_{k}^{j},\ \ r\in\mathbf{N},\tag{1}$$

where $_{i}x\left(r\right)$ is the element of sequence $_{i}x$; λ is a coefficient of importance of the relation between the upper and the lower levels. Finally, the sequence $_{i}x$ is as follows,

$$\begin{aligned}{}_{i}x&=\left({}_{i}x(1),\,{}_{i}x(2),\cdots\right)\\&=\left({}_{i}^{T}w_{1}^{1},\,{}_{i}^{T}w_{2}^{1},\cdots,\,\lambda\,{}_{i}^{T}w_{1}^{2},\,\lambda\,{}_{i}^{T}w_{2}^{2},\cdots,\lambda^{j-1}\,{}_{i}^{T}w_{1}^{m},\lambda^{j-1}\,{}_{i}^{T}w_{2}^{m},\cdots\right),0\le i\le l,\,0\le j\le m.\end{aligned}$$

Third Step. Construct the grey relational matrix

This step constructs the grey relational matrix. Consider a sequence, which is the referential sequence, for comparison to other sequences. Let $_{i}x$ be the referential sequence, where $_{i}x\left(r\right)$ is an element of the referential sequence, $i=1,2,\cdots,l;$ $r=1,2,\cdots,n$, $_{q}x$ is the sequence for comparison, and $_{q}x\left(r\right)$ is an element of the sequence for comparison, $q=1,2,\cdots,l$. The grey relational grade between $_{i}x$ and $_{q}x$ is calculated as follows [1,2,4,8].

$$\Gamma_{iq}=\frac{\Delta_{min}+\Delta_{max}}{\Delta+\Delta_{max}},\tag{2}$$

where, $\Delta=\frac{1}{n}\sum_{r=1}^{n}\Delta_{iq}$,$\Delta_{iq}\left(r\right)=\left|{}_{i}x\left(r\right)-{}_{q}x\left(r\right)\right|$, $\Delta_{min}=\min_{\forall i}\min_{\forall q}\Delta_{iq}\left(r\right)$, and $\Delta_{max}=\max_{\forall i}\max_{\forall q}\Delta_{iq}\left(r\right)$.

An $l\times l$ matrix can be determined by calculating all grey relational grades. The matrix is the grey relational matrix, R [1,2]:

$$R=\left[\Gamma_{iq}\right]_{l\times l}=\begin{bmatrix}\Gamma_{11}&\Gamma_{12}&\cdots&\Gamma_{1l}\\\Gamma_{21}&\Gamma_{22}&\cdots&\Gamma_{2l}\\\vdots&\vdots&\ddots&\vdots\\\Gamma_{l1}&\Gamma_{l2}&\cdots&\Gamma_{ll}\end{bmatrix}_{l\times l}.$$

Fourth Step. Calculate global grey relational grades

After the grey relational matrix, R, is obtained, λ_{max} of matrix R can be calculated by the eigenvalue technique, and then the priority eigenvector, v, of R, associated with λ_{max} can be determined, where the elements of v are the global grey relational grades for all sequences. All elements of v are ranked and the maximum element of v can be determined. The maximum element of v represents the optimal hierarchy structure [1,2].

3. Illustrative Example

The hierarchical structure schemas for teacher research performance are assumed to be represented by $_{1}H$ to $_{5}H$ as follows. See Figs. 1 to 5 [3].

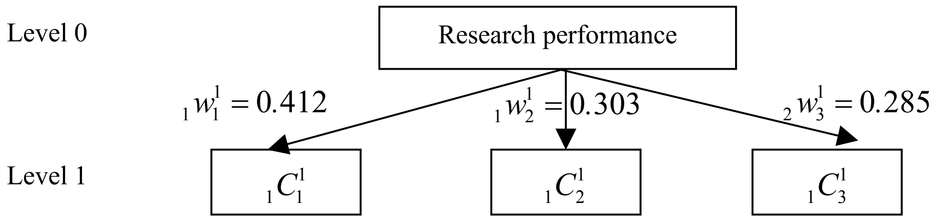

Figure 1. Hierarchy Structure $_1H$

where, $_1C_1^1$ refers to the paper; $_1C_2^1$ means the plan, and $_1C_3^1$ means the award.

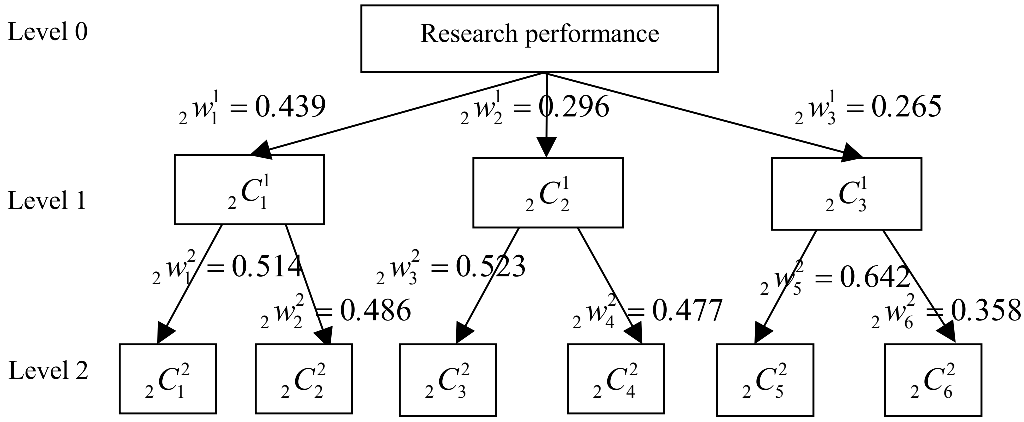

Figure 2. Hierarchy Structure $_2H$

where, $_2C_1^1$ represents the paper; $_2C_2^1$ means the Plan; $_2C_3^1$ refers the award; $_2C_1^2$ the amount of the dissertation; $_2C_2^2$ quality of the dissertation; $_2C_3^2$ the amount of the plan; $_2C_4^2$ quality of the plan; $_2C_5^2$ the amount of the award ,and $_2C_6^2$ quality of the award.

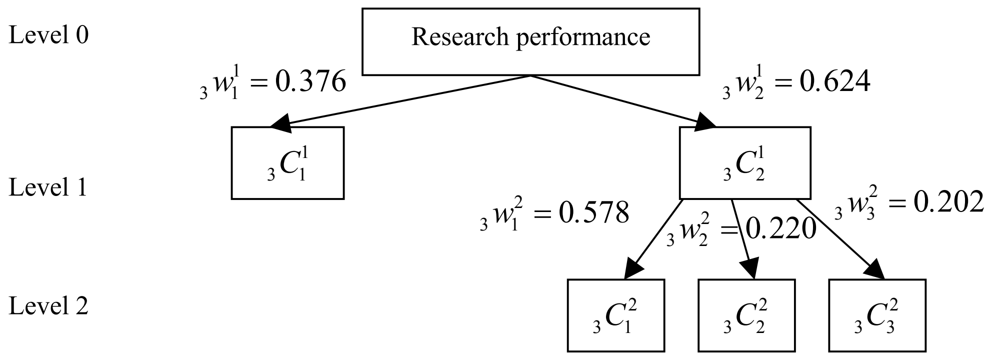

Figure 3. Hierarchy Structure $_3H$

where, $_3C_1^1$ refers to the quality; $_3C_2^1$ means the quantity; $_3C_1^2$ the amount of dissertation; $_3C_2^2$ the amount of plan, and $_3C_3^2$ the amount of award.

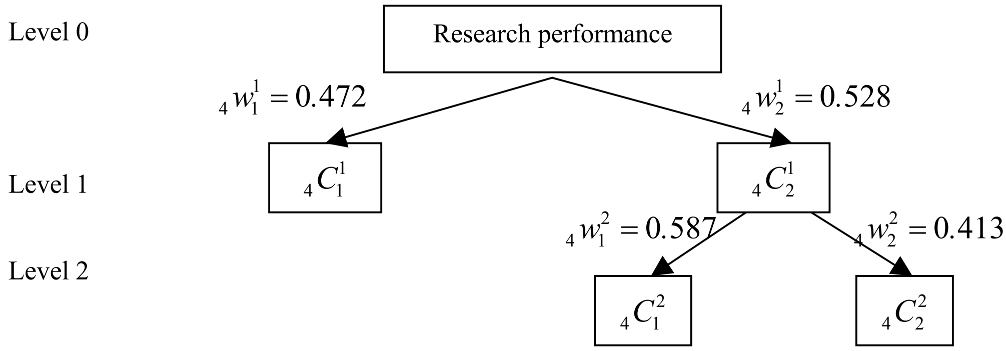

Figure 4. Hierarchy Structure $_4H$

where, $_4C_1^1$ represents the paper; $_4C_2^1$ refers to non-paper; $_4C_1^2$ the plan, and $_4C_2^2$ the award.

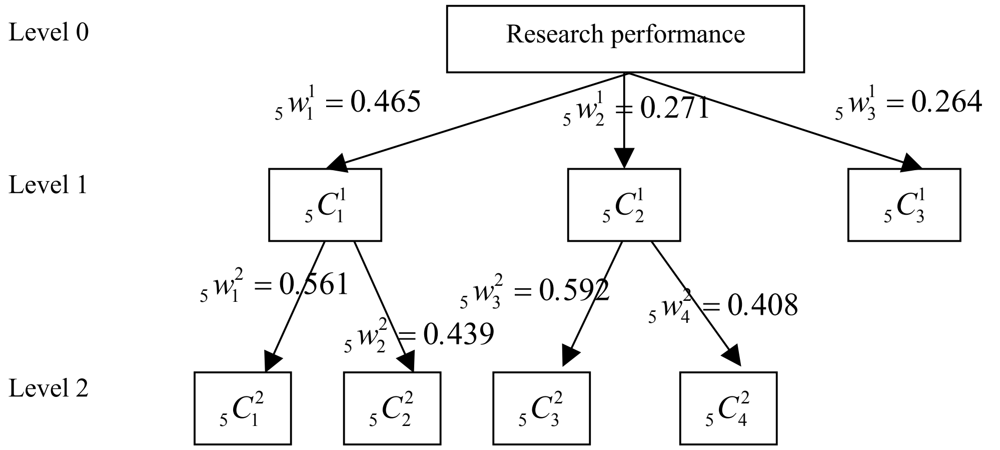

Figure 5. Hierarchy Structure $_5H$

where, $_5C_1^1$ refers to the paper; $_5C_2^1$ represents the Plan; $_5C_3^1$ means the award; $_5C_1^2$ the amount of dissertation; $_5C_2^2$ quality of dissertation; $_5C_3^2$ the amount of plan, and criteria $_5C_4^2$ quality of plan.

The novel algorithm determines the optimal hierarchical structure as follows.

First Step. Using set form to describe hierarchy structure.

Figs 1 to 5, sets can be obtained to describe the hierarchy structure $_1H,\ _2H,\ _3H,\ _4H,\ _5H$, as follows:

$$_1H = \left\{ {}_1C_1^1,\ {}_1C_2^1,\ {}_1C_3^1 \right\}$$

$$_2H = \left\{ {}_2C_1^1,\ {}_2C_2^1,\ {}_2C_3^1,\ {}_2C_1^2,\ {}_2C_2^2,\ {}_2C_3^2,\ {}_2C_4^2,\ {}_2C_5^2,\ {}_2C_6^2 \right\}$$

$$_3H = \left\{ {}_3C_1^1,\ {}_3C_2^1,\ {}_3C_1^2,\ {}_3C_2^2,\ {}_3C_3^2 \right\}$$

$$_4H = \left\{ {}_4C_1^1,\ {}_4C_2^1,\ {}_4C_1^2,\ {}_4C_2^2 \right\}$$

$$_5H = \left\{ {}_5C_1^1,\ {}_5C_2^1,\ {}_5C_3^1,\ {}_5C_1^2,\ {}_5C_2^2,\ {}_5C_3^2,\ {}_5C_4^2 \right\}$$

Second Step. Using sequence described set from of hierarchy structure

Using sets $_1H,\ _2H,\ _3H,\ _4H,\ _5H$ can be yield the macro hierarchical structure TH as

$$
\begin{aligned}
{}^{T}H &= {}_{1}H \cup {}_{2}H \cup \cdots \cup {}_{l}H \\
&= \left\{ {}_{1}C_1^1,\ {}_{1}C_2^1,\ {}_{1}C_3^1 \right\} \cup \left\{ {}_{2}C_1^1,\ {}_{2}C_2^1,\ {}_{2}C_3^1,\ {}_{2}C_1^2,\ {}_{2}C_2^2,\ {}_{2}C_3^2,\ {}_{2}C_4^2,\ {}_{2}C_5^2,\ {}_{2}C_6^2 \right\} \\
&\quad \cup \left\{ {}_{3}C_1^1,\ {}_{3}C_2^1,\ {}_{3}C_1^2,\ {}_{3}C_2^2,\ {}_{3}C_3^2 \right\} \cup \left\{ {}_{4}C_1^1,\ {}_{4}C_2^1,\ {}_{4}C_1^2,\ {}_{4}C_2^2 \right\} \\
&\quad \cup \left\{ {}_{5}C_1^1,\ {}_{5}C_2^1,\ {}_{5}C_3^1,\ {}_{5}C_1^2,\ {}_{5}C_2^2,\ {}_{5}C_3^2,\ {}_{5}C_4^2 \right\} \\
&= \left\{ {}^{T}C_1^1,\ {}^{T}C_2^1,\ {}^{T}C_3^1,\ {}^{T}C_4^1,\ {}^{T}C_5^1, \right. \\
&\quad \left. {}^{T}C_1^2,\ {}^{T}C_2^2,\ {}^{T}C_3^2,\ {}^{T}C_4^2,\ {}^{T}C_5^2,\ {}^{T}C_6^2,\ {}^{T}C_7^2,\ {}^{T}C_8^2,\ {}^{T}C_9^2,\ {}^{T}C_{10}^2,\ {}^{T}C_{11}^2 \right\}
\end{aligned}
$$

Next, the hierarchy structures ${}_{1}H,\ {}_{2}H,\ {}_{3}H,\ {}_{4}H,\ {}_{5}H$ are indicated the macro hierarchical structure as,

$$
\begin{aligned}
{}_{i}^{T}H &= \left\{ {}_{i}^{T}C_1^1,\ {}_{i}^{T}C_2^1,\ {}_{i}^{T}C_3^1,\ {}_{i}^{T}C_4^1,\ {}_{i}^{T}C_5^1, \right. \\
&\quad \left. {}_{i}^{T}C_1^2,\ {}_{i}^{T}C_2^2,\ {}_{i}^{T}C_3^2,\ {}_{i}^{T}C_4^2,\ {}_{i}^{T}C_5^2,\ {}_{i}^{T}C_6^2,\ {}_{i}^{T}C_7^2,\ {}_{i}^{T}C_8^2,\ {}_{i}^{T}C_9^2,\ {}_{i}^{T}C_{10}^2,\ {}_{i}^{T}C_{11}^2 \right\},\ 0 \le i \le 5.
\end{aligned}
$$

Then,

$$
\begin{aligned}
{}_{i}^{T}H' &= \left\{ {}^{T}w_1^1,\ {}^{T}w_2^1,\ {}^{T}w_3^1,\ {}^{T}w_4^1,\ {}^{T}w_5^1, \right. \\
&\quad \left. {}^{T}w_1^2,\ {}^{T}w_2^2,\ {}^{T}w_3^2,\ {}^{T}w_4^2,\ {}^{T}w_5^2,\ {}^{T}w_6^2,\ {}^{T}w_7^2,\ {}^{T}w_8^2,\ {}^{T}w_9^2,\ {}^{T}w_{10}^2,\ {}^{T}w_{11}^2 \right\},\ 0 \le i \le 5.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
{}_{1}^{T}H' = \{ & 0.412,\ 0.303,\ 0.285,\ 0.000,\ 0.000,\ 0.000, \\
& 0.000,\ 0.000,\ 0.000, 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000, 0.000,\ 0.000 \} \\
{}_{2}^{T}H' = \{ & 0.439,\ 0.296,\ 0.265,\ 0.000,\ 0.000, 0.000, \\
& 0.514,\ 0.486,\ 0.523,\ 0.477,\ 0.642,\ 0.358,\ 0.000,\ 0.000,\ 0.000,\ 0.000, 0.000 \} \\
{}_{3}^{T}H' = \{ & 0.000,\ 0.000,\ 0.000,\ 0.376,\ 0.624, 0.000, \\
& 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000, 0.000,\ 0.578,\ 0.220,\ 0.202,\ 0.000,\ 0.000 \} \\
{}_{4}^{T}H' = \{ & 0.472,\ 0.000,\ 0.000,\ 0.000,\ 0.000, 0.528, \\
& 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.587, 0.413 \} \\
{}_{5}^{T}H' = \{ & 0.465,\ 0.271,\ 0.264, 0.000,\ 0.000,\ 0.000, \\
& 0.561,\ 0.439,\ 0.592, 0.408,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000, 0.000 \}
\end{aligned}
$$

Appling Eq. (1) and $\lambda = 1/2$ can be yield ${}_{1}x,\ {}_{2}x,\ {}_{3}x,\ {}_{4}x,\ {}_{5}x$,

$$
\begin{aligned}
{}_1x = (&0.412, 0.303, 0.285, 0.000, 0.000, 0.000,\\
&0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)\\
{}_2x = (&0.439, 0.296, 0.265, 0.000, 0.000, 0.000,\\
&0.275, 0.243, 0.262, 0.239, 0.321, 0.179, 0.000, 0.000, 0.000, 0.000, 0.000)\\
{}_3x = (&0.000, 0.000, 0.000, 0.376, 0.624, 0.000,\\
&0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.289, 0.110, 0.101, 0.000, 0.000)\\
{}_4x = (&0.472, 0.000, 0.000, 0.000, 0.000, 0.528,\\
&0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.294, 0.207)\\
{}_5x = (&0.465, 0.271, 0.264, 0.000, 0.000, 0.000,\\
&0.281, 0.220, 0.296, 0.204, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)
\end{aligned}
$$

Third Step. Constructing grey relational matrix

Using Eq. (2) and ${}_1x,\ {}_2x,\ {}_3x,\ {}_4x,\ {}_5x$ yields matrix R :

$$
R = \begin{bmatrix} 1.000 & 0.872 & 0.809 & 0.864 & 0.906 \\ 0.872 & 1.000 & 0.725 & 0.773 & 0.941 \\ 0.809 & 0.726 & 1.000 & 0.780 & 0.752 \\ 0.864 & 0.773 & 0.780 & 1.000 & 0.805 \\ 0.906 & 0.941 & 0.752 & 0.805 & 1.000 \end{bmatrix}.
$$

Fourth Step. Calculating global grey relational grades

Finally, λ_{max} of the matrix R is calculated by the eigenvalue technique as $\lambda_{max} = 4.295$, and the priority eigenvector, v, of R, associated with λ_{max}, is,

$$
v = \begin{bmatrix} 0.464 & 0.450 & 0.422 & 0.439 & 0.460 \end{bmatrix}^{\mathrm{T}}.
$$

Ranking all elements of v yields, $0.464 > 0.460 > 0.450 > 0.439 > 0.422$. Consequently, the best hierarchy structure is ${}_1H$, followed by ${}_5H,\ {}_2H,\ {}_4H,\ {}_3H$ in order.

4. Conclusions

This study develops a novel algorithm that selects an optimal hierarchy structure of AHP model. Results in this study can be used to select an optimal hierarchy whit multiple hierarchical structure schema of AHP.

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