Valuing Life Insurance Contracts with Guaranteed Returns

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Abstract

In general, life insurance liabilities tend to have several embedded options. In this article we model life insurance contracts as contracts that guarantee a minimum return and that allow for early surrender. It turns out that other features of the contract, for instance whether it might be optimal to surrender it prior to maturity, depend heavily on the exact way in which this minimum return is being guaranteed.

1. Introduction

Since [1] numerous publications have appeared on the valuation of life insurance liabilities and are often based on the martingale pricing theory ([2], [3]). So for instance [1], [4], [5], [6], [7] and [8] consider the valuation of life insurance contracts that are directly or indirectly linked to an underlying asset which is being modelled as a geometric Brownian motion and [9], [10], [11] and [12] value life insurance contracts under stochastic interest rates.

Most of the authors concentrate on the modelling and valuation of the maturity guarantee which can be found in life insurance contracts, varying from the plain vanilla unit-linked contract in [1] to exotic unit-linked contracts, for instance contracts of which the pay-off depends on the value of two assets, in [6]. [8] and [13] take into account solvency margins and a compounding guaranteed return, which results in a type of contract of which the pay-off is highly path dependent.

In this paper, we will compare two different types of contracts by valuing their market-based single premiums. With respect to the pricing of life insurance contracts, mortality risk and financial risk should be treated simultaneously. The mortality risk however is regarded as an unsystematic risk so that we will solely discuss the market-based value. The two contracts we will focus on are: on one hand the contract that guarantees a minimum pay-off at maturity and on the other hand a contract that guarantees a minimum return over each of a number of sub-periods. We will allow for both contracts to be surrendered or ended prior to maturity at a number of preset dates. That is, the contracts will be Bermudan. We will show that despite the similarities between these two contracts, there are important differences with respect to their price and the optimal surrender behavior.

2. Two Types of Guaranteed Returns

In this section, we will present the two different contracts. We will compare their pay-offs at maturity and whether or not surrendering before maturity can be optimal. We will make no assumptions about the dynamics of the underlying asset or about the dynamics of the term-structure of interest rates. The first contract to be discussed is the contract guaranteeing a minimum pay-off at maturity.

2.1. The Maturity Guarantee

In [13] Grosen and Løchte Jørgensen model a single-premium contract by means of an American version of the Brennan and Schwartz ([1]) model, with the exercise price being an increasing function of time. We will only allow the put option to be exercised at a number of preset surrender dates $(t_i)_{i=1}^{n-1}$, with: $t_0 < \ldots < t_i < \ldots < t_{n-1} < t_n = T$. That is, at every surrender date t_i the policyholder has the possibility to either hold the contract until t_{i+1} or end the contract and walk away with the surrender value given by:

$$D \max\left(\frac{S(t_i)}{S(t_0)}, e^{r_G(t_i - t_0)}\right), \tag{1}$$

with *D* the amount that was initially invested in the underlying asset *S(t)*, the face value of the contract. As such, the policyholder holds a Bermudan put option on the return realized by the asset *S(t)* since time t_0, with maturity date *T* and exercise price $e^{r_G(t_i - t_0)}$ at every surrender date t_i. If the policyholder does not surrender the contract at any of the surrender dates, he receives the following pay-off at t_n:

$$D \max\left(\frac{S(t_n)}{S(t_0)}, e^{r_G(t_n - t_0)}\right). \tag{2}$$

We see that with this type of contract the policyholder is guaranteed the minimum return r_G over the actual lifetime of the contract. That is, whether the contract is ended prior to maturity or not, the minimum return paid out to the policyholder will never be less than r_G .

Theorem 1

For all of the *n-1* surrender dates t_i , *i=1,...n-1*,

$$\frac{S(t_i)}{S(t_0)} < e^{r_G(t_i - t_0)} \tag{3}$$

is a necessary condition in order for surrendering the contract at t_i to be optimal.

Proof

If the policyholder surrenders the contract at t_i and the above condition is not met, he receives the pay-off:

$$D\frac{S(t_i)}{S(t_0)}. \tag{4}$$

This amount can be invested in the underlying asset *S(t)* and will give rise to the following pay-off at maturity:

$$D\frac{S(t_n)}{S(t_0)}. \tag{5}$$

If the contract has not been ended, the policyholder will receive the following amount at maturity:

$$D\max\left(\frac{S(t_n)}{S(t_0)}, e^{r_G(t_n - t_0)}\right). \tag{6}$$

It is clear that this will always be at least as much as the amount given in formula 5.

Note that condition 3 is a necessary but not a sufficient condition for ending the contract to be optimal. We will illustrate that it can be optimal to surrender the contract at a surrender date.

EXAMPLE 1

Suppose we are dealing with a contract with two sub-periods, that is with one possible surrender date, and that *D=1*. If the policyholder does not surrender the contract at t_1 the pay-off at maturity, at t_2 , is given by:

$$\max\left(\frac{S(t_2)}{S(t_0)}, e^{r_G(t_2 - t_0)}\right). \tag{7}$$

Suppose that $S(t_1)/S(t_0) < e^{r_G(t_1 - t_0)}$. If the policyholder surrenders the contract at t_1 the pay-off is:

$$e^{r_G(t_1 - t_0)}. \tag{8}$$

If the policyholder does not surrender the contract, he holds a contract worth:

$$
\begin{aligned}
&\mathrm{P}(t_1, t_2)\, \mathrm{E}^T \left[\max\left(\frac{S(t_2)}{S(t_0)}, e^{r_G (t_2 - t_0)} \right) \Big|\, \mathcal{F}_{t_1} \right] \\
&= \mathrm{P}(t_1, t_2) \frac{S(t_1)}{S(t_0)} \mathrm{E}^T \left[\max\left(\frac{S(t_2)}{S(t_0)}, e^{r_G (t_2 - t_0)} \frac{S(t_0)}{S(t_1)} \right) \right] \\
&= \mathrm{P}(t_1, t_2) \left(e^{r_G (t_2 - t_0)} + \frac{S(t_1)}{S(t_0)} \mathrm{E}^T \left[\max\left(\frac{S(t_2)}{S(t_1)} - e^{r_G (t_2 - t_0)} \frac{S(t_0)}{S(t_1)}, 0 \right) \right] \right), \qquad (9)
\end{aligned}
$$

with $\mathrm{P}(t_1, t_2)$ the value at time t_1 of a zero-coupon bond with maturity date t_2. In the above equality, the expectation is taken under the *T*-Forward measure. If r_G is small enough, for instance r_G is smaller than the $(t_2 - t_1)$ - interest rate at t_1, that is:

$$
r_G < -\frac{\ln(\mathrm{P}(t_1, t_2))}{t_2 - t_1}, \qquad (10)
$$

then the last line in the above equality will be smaller than $e^{r_G (t_1 - t_0)}$ for small enough $S(t_1)/S(t_0)$. That is, it will be optimal to surrender the contract when the return realized over the first sub-period is low enough.

2.2. The Compounding Guarantee

With this type of contract a minimum rate r_G is guaranteed over the *n* sub-periods $[t_{k-1}, t_k]$, *k=1,...,n*, and the pay-off at maturity is given by:

$$
D \prod_{k=1}^{n} \max\left(\frac{S(t_k)}{S(t_{k-1})}, e^{r_G (t_k - t_{k-1})} \right), \qquad (11)
$$

where *D* is again the face value of the contract. The policyholder can surrender the contract at any of the surrender dates t_i, in which case the following amount will be paid:

$$
D \prod_{k=1}^{i} \max\left(\frac{S(t_k)}{S(t_{k-1})}, e^{r_G (t_k - t_{k-1})} \right). \qquad (12)
$$

This contract also guarantees a minimum return r_G, just like the previous one, (and it can be surrendered at any of the surrender dates t_i). The difference with the previous type of contract is, that the minimum return r_G is not only guaranteed over the maturity of the contract, but over each of the sub-periods $[t_{i-1}, t_i]$, *i=1,...,n*. If we compare formula 12 with formula 6, we see at every surrender date t_i, as well as at maturity, that the pay-off of the Maturity Guarantee contract will always be lower than or equal to the pay-off of the Compounding Guarantee contract. The equality will only occur when the return realized by the underlying asset *S(t)* is either smaller than r_G over every sub-period $[t_{i-1}, t_i]$ or greater than r_G over every sub-period $[t_{i-1}, t_i]$. If neither of these two extreme scenarios is realized, the pay-off of the Compounding Guarantee contract will dominate the one of the Maturity Guarantee contract.

Theorem 2

For the Compounding Guarantee contract we have the following result:
In case of a Compounding Guarantee contract it is never optimal to surrender the contract before maturity.

Proof

Trivial.

This means that even if the policyholder is allowed to surrender the contract prior to maturity, this need not to be taken into account when we value this contract.

2.3. Comparing the Two Contracts

The two contracts are rather similar in the sense that they both guarantee a minimum return on a nominal amount D investment in an underlying asset $S(t)$. In one case the minimum return applies over the entire maturity and the other contract guarantees a minimum return over every sub-period. Both contracts allow for early surrender at a number of preset dates. Despite these similarities, the two contracts are not really inter-changeable. One example is given by the fact that it might be optimal to surrender the first type of contract before maturity, whereas this will never be optimal for the second one.

This last difference has an important consequence for the reserving for the two types of contracts. In case of the Compounding Guaranteed return contract an initial reserve equal to the value at inception of the pay-off at maturity will constitute a prudent reserve. That is, for reserving purposes one can simply forget that the contract can be surrendered prior to maturity, since doing so will always be non-optimal. In case of the Maturity Guarantee contract the same reserving standards would lead to reserves being too low, because the possibility to surrender the contract before maturity has a certain value in itself, and this is not taken into account by this reserving method.

A second issue is the fact that the pay-offs and as such prices of the two contracts are ordered as given by the following theorem:

Theorem 3

If a Compounding Guaranteed return contract and a Maturity Guaranteed return contract have the same guaranteed return and surrender dates, then the Compounding Guaranteed return contract will be worth more then the Maturity Guaranteed return contract at every surrender.

Proof

The result follows immediately from the fact that at every surrender date, as well as at maturity, the pay-off of the Maturity Guarantee contract is dominated by the pay-off of the Compounding Guaranteed return contract.

As such we have that the contract for which it is not optimal to be surrendered prior to maturity dominates the contract that might be surrendered. That is, the European style product dominates the Bermudan one.

2.4. The Role of the Guaranteed Return

Since we work in a stochastic investment environment, the price of either of these two contracts can not be obtained by discounting at the guaranteed return. However, clearly the guaranteed return r_G will still play a role in setting the price of these contracts. Moreover, the guaranteed return will have an effect on the value of these contracts through two different mechanisms. This effect takes the following two-step form.

Step 1

Like in the case of traditional life insurance contracts, these two contracts offer a minimum pay-off for every invested Euro. As such, it would be interesting to look to these contracts from the opposite perspective. How much does one need to pay initially in order to obtain at least a given amount at maturity ? Or, what value should the nominal amount D have, in order to obtain at least one Euro at maturity? For both types of contracts the nominal amount needed to effectively guarantee one Euro at maturity is given by:

$$D = e^{-r_G T}, \tag{13}$$

with T the maturity of the contract. That is, the guaranteed return, sets the nominal amount of the contract. The nominal amount being equal to the minimum pay-off at maturity discounted at the guaranteed return.

Step 2

Having determined the value of D, one can calculate the price of the two contracts for this value of D. As such, we see that the guaranteed return r_G has a double effect on the value of a contract that guarantees a certain minium pay-off at maturity. First it determines the nominal amount D that is needed to effectively guarantee this minimum pay-off at maturity. Second, the guaranteed return r_G will have an effect, together with other parameters, on the price of the contract with this D as its nominal amount.

Since for a given nominal amount D the price of either of the two contracts is increasing in r_G these two effects will partially compensate for each other.

3. Valuation of the Maturity Guarantee

3.1. General Discussion

We first will introduce the concept of a compound call option. A compound call option (of order 2) is a call option on a call option, i.e. a call option of which the underlying is itself again a call option ([14], [15] and [16]). For instance, consider a contract that entitles one to the following pay-off at t_1 :

$$\max \{C(t_1, S(t_1), t_2, K_2), K_1\} \tag{14}$$

That is, at t_1, the investor receives the maximum of the amount K_1 and the value of a European call on the asset S with exercise date and price given by t_2 and K_2 respectively. In this case the investor holds a call, exercisable at t_1, on a underlying call which in turn is exercisable at t_2. We can generalize this to a compound call of order *i* (with exercise date and price given by t_1 and K_1) with an underlying call of order *i-1* (with exercise date and price given by t_2 and K_2), which itself is a call on a call of order *i-2, ...* until we reach the final underlying European call (with exercise date and price given by t_i and K_i). We denote the price at time t_0 of such call of order *i* by:

$$C^{(i)}\left(t_0, S(t_0); (t_j, K_j)_{j=1}^{i}\right), \tag{15}$$

with (t_j, K_j) the exercise date and exercise price of the call of order *i - j+1*, underlying the call of order *i*. Having introduced the concept of compound options, we obtain the following theorem that enables us to price the life insurance liabilities.

Theorem 4

If at any date t_i *(i=0,1,...,n-1)* the value of the future payments, i.e. payments from t_{i+1} onwards, of the life insurance contract that the insured will receive when he exercises the surrender option rationally, is given by:

$$C^{(n-i)}\left(t_i, S(t_i); (t_{n-k}, \Delta_{n-k})_{k=0}^{i-1}\right) + K(t_{n-i+1})e^{-(t_{n-i+1}-t_{n-i})r} \tag{16}$$

with:

$$\Delta_{n-k} = K(t_{n-k}) - K(t_{n-k+1})e^{-(t_{n-k+1}-t_{n-k})r}, \quad k = 1,..., n-1 \tag{17}$$

$$\Delta_n = K(t_n), \tag{18}$$

the price at inception of the contract as described above is given by:

$$C^{(n)}\left(t_0, S(t_0); (t_{n-k}, \Delta_{n-k})_{k=0}^{n-1}\right) + K(t_1)e^{-(t_1-t_0)r}. \tag{19}$$

Proof

The proof is given by induction.

At t_{n-1}:

If the insured has decided not to walk away at t_{n-1}, he holds a contract that guarantees the payment of $B(t_n) = \max\left\{S(t_n), S(t_0)\mathrm{e}^{(t_n-t_0)r_G}\right\}$ at t_n. As such the value of the contract is given by:

$$C(t_{n-1}, S(t_{n-1}), t_n, K(t_n)) + K(t_n)\mathrm{e}^{-(t_n-t_{n-1})r}, \tag{20}$$

with $K(t_n) = K(T)$, the maturity guarantee.
This proves formula (19) for *n=1*.

At t_{n-2}:

If at t_{n-2} the insured has not exercised his surrender option, he holds a contract that guarantees him of a payment at t_{n-1} given by:

$$\max\left\{C(t_{n-1}, S(t_{n-1}), T, K(t_n)) + K(t_n)\mathrm{e}^{-(t_n-t_{n-1})r}, K(t_{n-1})\right\}. \tag{21}$$

That is, at t_{n-1} he will receive the maximum of the value given by (20) and $K(t_{n-1})$, the surrender value at t_{n-1}. The value of the contract equals

$$
\begin{aligned}
&\mathrm{e}^{-(t_{n-1}-t_{n-2})r}\,\mathrm{E}^{Q}\left[\max\left\{C(t_{n-1},S(t_{n-1}),T,K(t_n))+K(t_n)\mathrm{e}^{-(t_n-t_{n-1})r},K(t_{n-1})\right\}\Big|_{\;t_{n-2}}\right]\\
&=\mathrm{e}^{-(t_{n-1}-t_{n-2})r}\,\mathrm{E}^{Q}\left[\max\left\{C(t_{n-1},S(t_{n-1}),T,K(t_n))-(K(t_{n-1})-K(t_n)\mathrm{e}^{-(t_n-t_{n-1})r}),0\right\}\Big|_{\;t_{n-2}}\right]\\
&\quad+\mathrm{e}^{-(t_{n-1}-t_{n-2})r}K(t_{n-1})\\
&=C\circ C(t_{n-2},S(t_{n-2});t_{n-1},\Delta_{n-1};t_n,\Delta_n)+\mathrm{e}^{-(t_{n-1}-t_{n-2})r}K(t_{n-1}).
\end{aligned}
\tag{22}
$$

Formula (22) shows that the value of the contract consists of two parts. The first term on the right side of the second equality is the value of a compound call, the second term is an investment in the mo ney market account.

At t_{n-i}:
The induction hypothesis is used and completely analogous, one obtains that at any surrender date t_{n-i}, the value of the future payments when the insured has decided not to end the contract is given by:

$$
\begin{aligned}
&\mathrm{e}^{-(t_{n-i+1}-t_{n-i})r}\,\mathrm{E}^{Q}\Big[\max\Big\{C^{(i-1)}\left(t_{n-i+1},S(t_{n-i+1});(t_{n-k},\Delta_{n-k})_{k=0}^{i-1}\right)\\
&\qquad+K(t_{n-(i-1)+1})\mathrm{e}^{-(t_{n-(i-1)+1}-t_{n-(i-1)})r},K(t_{n-(i-1)})\Big\}\Big|_{\;t_{n-i}}\Big].
\end{aligned}
\tag{23}
$$

with $C^{(i-1)}\left(t_{n-i+1},S(t_{n-i+1});(t_{n-k},\Delta_{n-k})_{k=0}^{i-1}\right)$ the price at t_{n-i+1} of a compound call of order *i-1* of which by induction the *i-1* exercise dates and prices are given by t_{n-k} and Δ_{n-k} with:

$$
\begin{aligned}
\Delta_{n-k}&=K(t_{n-k})-K(t_{n-k+1})\mathrm{e}^{-(t_{n-k+1}-t_{n-k})r}, \qquad k=1,\dots,i-1\\
\Delta_n&=K(t_n).
\end{aligned}
\tag{24}
$$

We now obtain that the value of the future payments is equal to:

$$
\begin{aligned}
&\mathrm{e}^{-(t_{n-i+1}-t_{n-i})r}\,\mathrm{E}^{Q}\Big[\max\Big\{C^{(i-1)}\left(t_{n-i+1},S(t_{n-i+1});(t_{n-k},\Delta_{n-k})_{k=0}^{i-1}\right)\\
&\quad-(K(t_{n-i-1})-K(t_{n-i+2})\mathrm{e}^{-(t_{n-i+2}-t_{n-i+1})r}),0\Big\}\Big|_{\;t_{n-i}}\Big]+K(t_{n-i+1})\mathrm{e}^{-(t_{n-i+1}-t_{n-i})r}\\
&=C^{(i)}(t_{n-i},S(t_{n-i});(t_{n-k},\Delta_{n-k})_{k=0}^{i-1})+K(t_{n-i+i})\mathrm{e}^{-(t_{n-i+1}-t_{n-i})r}.
\end{aligned}
\tag{25}
$$

In the next section we will obtain a semi-analytical expression for the price of the life insurance liabilities in case the underlying assets follow a geometric Brownian motion.

3.2. Valuation under the Black and Scholes ([17]) Assumptions

Consider a risky asset *S(t)* defined on a filtered probability space $(\Omega,\mathrm{P},(\ _t)_t)$ where the filtration $(\ _t)_t$ is generated by the process *S(t)* . Furthermore, we assume that *S(t)* satisfies:

$$
\frac{dS(t)}{S(t)}=m\,dt+s\;dB(t),
\tag{26}
$$

with $s>0$ and that there exists a riskless money market account in which money can be invested at a spot rate *r, r>0.*

Under these assumptions, we obtain the following result for the value of compound call options.

Theorem 5

In an economy as described above, the value of a Maturity Guaranteed contract (an *n*-fold compound call option) $C^{(n)}(t_0,S(t_0);(t_k,K_k)_{k=1}^{n})$ can be computed as:
for *n=1:*

$$
C^{(1)}\left(t_0,S(t_0);t_1,K_1\right)=S(t_0)N(d_1)-e^{-(t_1-t_0)r}K_1N(d_1\text{ - }s\sqrt{t_1-t_0})
\tag{27}
$$

and for *n>1:*

$$C^{(n)}\left(t_0, S(t_0); (t_k, K_k)_{k=1}^{n}\right)=$$

$$S(t_0)N_n(d_1^{(n)} + s\sqrt{t_1 - t_0}, ..., d_n^{(n)} + s\sqrt{t_n - t_0}; \Sigma^{(n,n)})$$

$$-\sum_{i=0}^{n-1} K_{n-i} e^{-r(t_{n-i}-t_0)} N_{n-i}(d_1^{(n)}, ..., d_{n\text{-}i}^{(n)}; \Sigma^{(n,n-i)}), \tag{28}$$

with for all $i=0,1,\ldots,n-1$:

$$d_{n-i} = \frac{\ln(S(t_0)/c_{n-i}) + (r - s^2/2)(t_{n-i} - t_0)}{s\sqrt{t_{n-i} - t_0}}, \tag{29}$$

the value for c_{n-i} in formula 29 given by the solution of the following equation:

$$C^{(i)}\left(t_{n-i}, c_{n-i}(t_{n-j}, K_{n-j})_{j=0}^{i-1}\right) = K_{n-i}, \tag{30}$$

and $N_{n-i}(x_1, ..., x_{n-i;}\Sigma^{(n,n-i)})$ the multivariate normal probability determined by the *(n-i)* tuple $(x_1, ..., x_{n-i})$ for a multivariate normal distribution with a vector of means equal to the null vector and a covariance (correlation) matrix$\Sigma^{(n,n-i)}$ (with $i=1,...,n-1$, the matrix$\Sigma^{(n,n-i)}$ being the sub-matrix of the *n-i* first rows and columns of matrix$\Sigma^{(n,n)}$) with the elements $(r_{kl}^{(n,n)})_{kl}$ of the matrix $\Sigma^{(n,n)}$ given by:

$$\begin{aligned} r_{kl}^{(n,n)} &= \sqrt{\frac{t_{k+i} - t_i}{t_{l+i} - t_i}} \quad \text{for } k < l \\ r_{kl}^{(n,n)} &= 1 \quad \text{for } k = l \\ r_{kl}^{(n,n)} &= \sqrt{\frac{t_{l+i} - t_i}{t_{k+i} - t_i}} \quad \text{for } k > l. \end{aligned} \tag{31}$$

Proof

For the proof of the above theorem we refer to [18].

Note that at inception the price of the contract does not depend on the value of the underlying asset *S(t)*. This is a result of the fact that under the above assumptions the future changes in *S(t)* are independent of the current value of *S(t)*. However, at any of the future surrender dates the value of the contract will clearly depend on the returns realized over the already elapsed sub-periods.

Some sensitivity analysis as generalization of the results obtained by [14] reveal the following results

$$\begin{aligned} \frac{C^{(n)}}{S} &= N_n(d_1^{(n)} + s\sqrt{t_1 - t_0}, ..., d_n^{(n)} + s\sqrt{t_n - t_0}; \Sigma^{(n,n)}) \\ \frac{C^{(n)}}{s^2} &= \frac{1}{2s}\sum_{j=0}^{n-1} \frac{1}{\sqrt{2p}} \mathrm{e}^{-\frac{1}{2}d_{j+1}^2} c_{n-j+2}\, \mathrm{e}^{-r(t_{j+1}-t_0)} N_{n-1}(.) \end{aligned} \tag{32}$$

as was shown by [19]. From these expressions, we observe that any increase in S and s^2 increases the expected pay-off to the option.

4. Valuation of the Compounding Guaranteed Return

Since the pay-off at maturity of this contract is given by formula 11 and we can always assume to be dealing with a European version of the contract, the price of the contract will be given by:

$$D\,\mathrm{E}^{Q}\mathop{}_{i=1}^{n}\max\left(\frac{S(t_i)}{S(t_{i-1})},e^{r_G(t_i-t_{i-1})}\right)\Big|S(t_0)\;.\tag{33}$$

Since in general the returns over the different sub-periods will not be independent, it might be hard, or even impossible to obtain an analytical expression for this expected value. We will now turn to the Black and Scholes case.

4.1. Valuation under the Black and Scholes Assumptions

If we assume that *S(t)* satisfies:

$$\frac{dS(t)}{S(t)}=m\;dt+s\;dB(t),\tag{34}$$

with $s>0$, and that there exists a riskless money market account in which money can be invested at a spot rate *r, r>0*, the following result can be obtained.

Theorem 6

Under the above assumptions the value of a compounding guaranteed return contract as described above is given by:

$$D\mathop{}_{i=1}^{n}\left(N(d_i)+e^{-(r-r_G)(t_i-t_{i-1})}N\left(-d_i+s\sqrt{t_i-t_{i-1}}\right)\right),\tag{35}$$

with:

$$d_i=\frac{(r-r_G+s^2/2)(t_i-t_{i-1})}{s\sqrt{t_i-t_{i-1}}}.\tag{36}$$

Proof

Because of the independence of the returns over the different sub-periods, one has:

$$\begin{aligned}p&=D\,\mathrm{E}^{Q}\mathop{}_{i=1}^{n}\max\left(\frac{S(t_i)}{S(t_{i-1})},e^{r_G(t_i-t_{i-1})}\right)\Big|S(t_0)\\&=D\mathop{}_{i=1}^{n}\mathrm{E}^{Q}\max\left(\frac{S(t_i)}{S(t_{i-1})},e^{r_G(t_i-t_{i-1})}\right)\Big|S(t_0)\quad.\end{aligned}\tag{37}$$

The rest is a straightforward application of the Black & Scholes formula.

Again, as a consequence of the Black and Scholes assumptions, the price of the contract does not depend on the value of the underlying asset *S(t)* at inception.

Corollary

In case t_i-t_{i-1} is independent of *i*, say $t_i-t_{i-1}=t$, we obtain the value of the contract by:

$$D(N(d)+e^{-(r-r_G)t}N\left(-d+s\sqrt{t}\right))^n\tag{38}$$

with:

$$d=\frac{(r-r_G+s^2/2)t}{s\sqrt{t}}.\tag{39}$$

In this last case, it is interesting to go back to the discussion on the role of the guaranteed return. There, we already mentioned that the main purpose of including a guaranteed minimum return in an investment contract often is to make it possible to guarantee a minimum pay-off, that is an amount, at maturity. The value of the guaranteed return r_G then determines the nominal amount *D* of the contract. Let us consider a contract that guarantees a minimum pay-off of a

Euro at maturity. Let us further assume that the contract has n sub-periods of equal length t . In this case the initial nominal amount is given by:

$$e^{-r_G(t_n - t_0)} = e^{-n\, r_G\, t}, \tag{40}$$

with t_0 the starting date, and t_n the maturity date of the contract. As such, the value of this contract is given by:

$$\frac{\left(N(d) + e^{-(r-r_G)t}\, N\left(-d + s\sqrt{t}\right)\right)^n}{e^{r_G t}}. \tag{41}$$

Formula 41 is equivalent with:

$$v^n, \tag{42}$$

with:

$$v = e^{-r_G t}\left(N(d) + e^{-(r-r_G)t}\, N\left(-d + s\sqrt{t}\right)\right). \tag{43}$$

As such we see that under the Black & Scholes assumptions the Compounding Guaranteed return contract leads to discounting at a constant discount factor *v*, similar as in the classical actuarial approach. With a constant discount factor, we mean that this factor does not depend on the maturity of the contract. However, the main difference with the classical actuarial method is, that the discount factor v is no longer equal to $e^{-r_G t}$. Now, *v* is the product of $e^{-r_G t}$ with the factor $N(d) + e^{-(r-r_G)t} N(-d + s\sqrt{t})$, that takes into account the options embedded in the contract. With a specific choice of the value for the parameters, *v* becomes equal to one and in that case the contract is issued a pari, in the sense that the price is equal to the minimum amount guaranteed at maturity.

5. Numerical Examples

We will illustrate the results obtained under the Black & Scholes assumptions with some numerical examples.

5.1 The Maturity Guarantee

We consider a contract with a maturity of 20 years and three different surrender dates: $t_1 = 5, t_2 = 10, t_3 = 15$.

5.1.1. The Case of a Constant Nominal Amount

Assuming that the nominal amount *D*, to which the guarantee applies, is equal to 1 Euro, Table 1 illustrates how the price of the Maturity Guarantee contract changes as a function of the spot rate r_G and the guaranteed interest rate *r*. For this table σ is equal to 20%.

Table 1: The Combined Effect of the Guaranteed Return and the Spot Rate

		r				
		4.0%	5.0%	6.0%	7.0%	8.0%
	2.0%	1.17587	1.12034	1.07790	1.04607	1.02284
	3.0%	1.24790	1.17587	1.12034	1.07790	1.04607
r_G	4.0%	1.34528	1.24790	1.17587	1.12034	1.07790
	5.0%		1.34528	1.24790	1.17587	1.12034
	6.0%			1.34528	1.24790	1.17587

From the above table we observe that, under the Black and Scholes assumptions, the price of the Maturity Guarantee contract only depends on the difference between the spot rate and the guaranteed return (just as for the Compounding Guarantee contract).

From table 2 we obtain an idea of how the value of the contracts is affected by changes in the difference between the spot rate *r* and the guaranteed return r_G on one hand and the volatility σ of the underlying asset on the other.

Table 2: The Combined Effect of the Difference between the Spot Rate and the Guaranteed Return and the Volatility

		r-r_G				
		0.0%	1.0%	2.0%	3.0%	4.0%
	10%	1.17694	1.09709	1.04638	1.01527	0.99837
	15%	1.26268	1.17302	1.10970	1.06387	1.03174
σ	20%	1.34528	1.24790	1.17587	1.12034	1.07790
	25%	1.42385	1.32002	1.24119	1.17840	1.12835
	30%	1.49767	1.38839	1.30400	1.23543	1.17941

We see that, at inception, the value of the contract is always higher than the nominal amount, and that, as expected, it increases in the volatility of the underlaying asset. Secondly, we observe that the price of the contract is decreasing in the difference between the spot rate and the guaranteed return.

5.1.2. The Case of a Constant Minimum pay-off at Maturity

Assuming that the minimum amount guaranteed at maturity is held constant at 1 Euro, the nominal amount will depend on the guaranteed return r_G as follows:

$$D = \mathrm{e}^{-r_G T}, \qquad (44)$$

with T the maturity of the contract. Table 3 shows that in this case the value of such a contract, that is with the guaranteed minimum pay-off being constant, is a function of both the spot rate r and the guaranteed return r_G, and not only of the difference between these two. For table 3, ← is again equal to 0.20.

Table 3: The Combined Effect of the Guaranteed Return and the Spot Rate

		r				
		4.0%	5.0%	6.0%	7.0%	8.0%
	2.0%	0.79133	0.75396	0.72540	0.70398	0.68834
	3.0%	0.69093	0.65105	0.62031	0.59681	0.57918
r_G	4.0%	0.61397	0.56953	0.53665	0.51131	0.49194
	5.0%		0.50702	0.47032	0.44317	0.42224
	6.0%			0.41946	0.38910	0.36664

Table 3 also shows that an increase in the guaranteed return r_G leads to a decrease in the price of the contract. As such, although for a given nominal amount a higher value for r_G leads to an increase in the price, if we keep the guaranteed minimum pay-off constant, increasing r_G results in a lower price. If we now go back to section 2.4, we see that the first mechanism through which r_G affects the price of the contracts dominates.

Table 4 draws a picture of the sensitivity of this contract to the values of the volatility of the underlying asset and the difference between the spot rate and the guaranteed interest rate. Here, r_G is equal to 4%.

Table 4: The Combined Effect of the Spot Rate and the Volatility

		r				
		4.0%	5.0%	6.0%	7.0%	8.0%
σ	10%	0.53714	0.50070	0.47755	0.46336	0.45564
	15%	0.57627	0.53535	0.50645	0.48554	0.47087
	20%	0.61397	0.56953	0.53665	0.51131	0.49194
	25%	0.64983	0.60244	0.56646	0.53781	0.51496
	30%	0.68352	0.63364	0.59513	0.56383	0.53827

5.2 The Compounding Guarantee

We will give numerical results about the sensitivity of value of the Compounding Guarantee contract to the values of the different parameters. As in section 5.1, the contract has a maturity of 20 years and can be surrendered at three different surrender dates: $t_1 = 5, t_2 = 10, t_3 = 15$.

5.2.1 The Case of a Constant Nominal Amount

In this section we assume that the nominal amount D to which the guarantee applies, is equal to 1 Euro. Table 5 gives numerical results about the sensitivity of the value of the contract to changes in the volatility of the underlaying asset and the difference between the spot rate r and the guaranteed return r_G. From section 4 we know that the value of the contract only depends on the difference between these two returns, not on the value of each of them.

Table 5: The Combined Effect of the Difference between the Spot Rate and the Guaranteed Return and the Volatility

		σ				
		10.0%	15.0%	20.0%	25.0%	30.0%
r-r_G	0%	2.05577	2.78937	3.65759	4.64450	5.72368
	1%	1.86848	2.53193	3.31788	4.21153	5.18884
	2%	1.71342	2.31280	3.02491	3.83531	4.72185
	3%	1.58497	2.12588	2.77163	3.50760	4.31315
	4%	1.47857	1.96610	2.55214	3.22148	3.95464

5.2.2 The Case of a Constant Minimum pay-off at Maturity

Assuming that the minimum amount guaranteed at maturity is held constant at 1 Euro, table 6 and figure 1 show how the value of this contract depends on the values of the guaranteed return and the spot rate. In table 6, ← is set equal to 20%.

Table 6: The Combined Effect of the Guaranteed Return and the Spot Rate

		R			
		4.0%	5.0%	6.0%	7.0%
r_G	2.0%	2.03567	1.86523	1.71752	1.58923
	3.0%	1.83703	1.67482	1.53458	1.41306
	4.0%	1.66927	1.51424	1.38053	1.26493
	5.0%	1.52733	1.37851	1.25047	1.14006
	6.0%	1.40706	1.26358	1.14045	1.03453

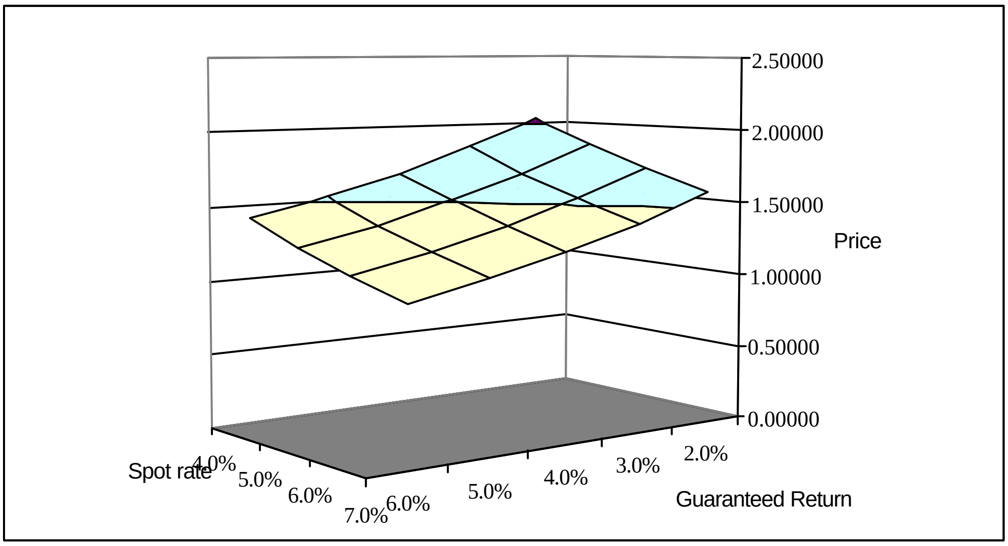

Figure 1: The Combined Effect of the Guaranteed Return and the Spot Rate

Table 7 and figure 2 show how changes in the value of the spot rate and the volatility affect the value of this contract.

Table 7: The Combined Effect of the Difference between the Spot Rate and the Guaranteed Return and the Volatility

		σ				
		10.0%	15.0%	20.0%	25.0%	30.0%
r	4.0%	0.93823	1.27303	1.66927	2.11969	2.61221
	5.0%	0.85275	1.15554	1.51424	1.92209	2.36812
	6.0%	0.78198	1.05553	1.38053	1.75039	2.15499
	7.0%	0.72336	0.97022	1.26493	1.60082	1.96847
	8.0%	0.67480	0.89730	1.16476	1.47024	1.80485

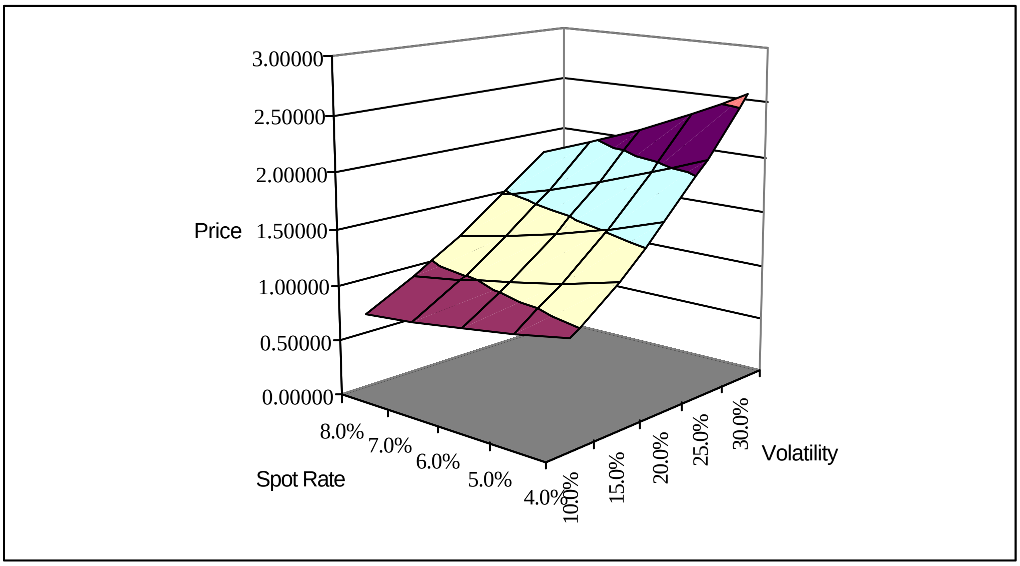

Figure 2: The Combined Effect of the Difference between the Spot Rate and the Guaranteed Return and the Volatility

Note that the above tables and figures illustrate theorem 3, that is the price of the Compounding Guarantee contract dominates the price of the Maturity Guarantee contract.

6. Conclusion

We showed that of the several embedded options that can be found in traditional with-profits life insurance contracts and unit-linked contracts, none can be modelled independently of one another. Especially, there turns out to be a close connection between the surrender feature and the precise way in which the guaranteed return has been modelled. Also reserving methods are affected by how the minimum return has been defined.

With respect to future research, it would be interesting to see how these results can be replicated under a more general model of the financial market, e.g. under a stochastic term-structure model. Secondly, it might be interesting to see what the implications would be of allowing the contracts to be sold for a (constant) periodic premium.

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