A MARKOV CHAIN MODEL OF SERIAL PRODUCTION SYSTEMS WITH REWORK

George C. Hadjinicola

Department of Public and Business Administration, School of Economics and Management,
University of Cyprus, Kallipoleos 75, P.O. Box 20537, CY 1678 Nicosia, Cyprus
Tel: +357-2-892473, Fax: +357-2-892460, Email: bageorge@ucy.ac.cy

Abstract

In this paper, we present a Markovian modeling framework that can describe any serial production system with rework. Under this framework, each production stage is represented by a state in the Markov chain. Absorbing states indicate the events of scrapping a product at a production stage or the completion of the finished product. Generalizable formulae for the final absorption probabilities are derived that represent: (1) the probability that an unfinished product is scraped at a certain production stage and (2) the yield of the system. We also derive various expected costs and quantities associated with all products ending in any absorbing state, as well as the equivalent costs and quantities for finished products. The applicability of our modeling framework is demonstrated in a real-life manufacturing environment in the food-packing industry. We show how the model can be used to assess the impact of changes in the transition probabilities on the various expected costs and quantities. The model can be used as a decision-making tool to assist management identify the production stage(s) that if improved, will yield the maximum benefit in terms of the costs and yield of the serial production system.

1. Introduction

Certain production systems follow a serial configuration. Examples of serial production systems include electronic and automotive assembly systems, semiconductor fabrication facilities, chemical processes, and food processing and packing systems. Serial production systems are in a sense simpler forms of multistage production systems. The majority of the literature on serial production systems has been on production control issues where, given the random yield at each production stage, the objective is to determine the optimal production policy that minimizes various costs. For example, Lee and Yano [20] developed an approach for production control in serial production systems where the yield at each production stage follows a distribution. Their objective was to determine in a single-period model, the optimal input quantity at each production stage to minimize the expected total cost, given a fixed and known demand for the single product produced by the serial production system. Tang [26] presents a discrete time model of a serial production system facing uncertainty in the output rate and demand, in order to examine the impact of these uncertainties on the production level and inventory at each production stage. Work on production planning in serial production systems has also been done by Kimura and Terada [16], Wein [27], Karimi [15], Akella et al. [2], Hendricks and McClain [13], Grosfeld-Nir and Robinson [10], Lee [19], and Hwang and Singh [14]. Denardo and Lee [6] provide a review on the control of serial production systems.

Yano [8] examined a two-stage serial production system where its processing and procurement times were stochastic. Yano presented a solution procedure that determined the optimal lead times which minimize the sum of inventory holding costs and job tardiness costs given a customer specified due-date. Optimal lead times in serial production systems were also examined by Gong et al. [9] and Mohan and Ritzman [22]. Other related work on serial production systems includes lot sizing in serial systems. Harrison and Lewis [11] and Yano and Lee [29] provide a review on the topic. Conway et al. [4] examine the role of work-in-process inventory in serial production systems. Ballou and Pazer [3] and Rebello et al. [23] examine inspection issues in serial production systems.

While considerable research has been done on production planning in serial production systems, the concept of rework in such production systems has not been adequately addressed. Recognizing this need, Lee and Yano [20] point that more research must be done on rework issues in serial production systems. The importance of rework is highlighted in a study by Fisher and Ittner [8] who examined the impact of product variety on the performance of a GM automobile plant. They found that in each month, on average, 1.9% of the finished products required major rework. They further showed that the percentage of products requiring rework had a significant impact on the overhead labor hours implying an increase of the average cost per unit.

Modeling work incorporating rework in serial production systems has been carried out by Akella et al. [2] and Wein [27]. Akella et al. modeled a discrete time, multiproduct, two-stage production system with rework, where test operations were performed at each production stage. The objective was to determine the optimal dispatch policy that minimizes the long-term expected cost which consisted of the sum of inventory holding, backlogging, and overtime costs over a finite horizon. Each production stage was characterized by its random yield and demand was also random. They proposed two approximate policies for the optimal dispatch policy. Wein [27] presented a production planning model for a serial production system where at each production stage, decisions must be made regarding the number of "good" products to be sent to the next stage, as well as the number of products to be sent for rework. Modeling approaches that considered rework include the work by Denardo and Tang [5], and Lee [18]. So and Tang [25] and Agnihothri and Kenett [1] consider rework issues in a single-stage production system. Sarkar and Zangwill [24] link cyclic queue systems and production systems with rework.

Rework in general, requires additional labor hours and perhaps the scraping of defective components which further results in the increase of the average unit cost. As such, one aspect of rework that the above studies did not address is the cost of rework and the impact that the yield of each production stage has on such cost. Our aim in this paper is to provide a modeling framework that address the following question: Given a production system where rework is allowed, which production stages should management try to improve in order to obtain the maximum benefits in terms of the yield and the costs associated with the entire production system?

In this paper we model a production system with rework as a first-order Markov chain where each production stage is represented by a transition state. Absorbing states, associated with each production stage, represent the termination of the production process in the events that an unfinished product is scraped or completed into a finished product. The model facilitates the derivation of formulae for the final absorption probabilities which represent: (1) the probability that an unfinished product is scraped at a certain production stage and (2) the probability that a product is completed, which is also the yield of the system. In addition, we derive various expected costs and quantities for the serial production system for all products ending in absorbing states. We also derive the equivalent formulae of the costs and quantities for finished products. The latter costs are of particular interest to the management of a firm operating a serial production system. Such expected costs and quantities include, for example, the time and cost spent on rework by each product, the number of production stages visited, and the time required by each product for its completion.

The Markov chain modeling framework is generalizable and can model any possible configuration of a serial production system with rework. This is accomplished by setting the appropriate transition probabilities to zero. The model can be used to assess the impact of changes in the transition probabilities (for products moving to the next or previous production stage) on the various expected costs and quantities mentioned above. This can be accomplished without performing simulations since closed form formulae have been derived. This provides insights as to where management should focus its efforts in order to improve the system. It allows through a sensitivity analysis to see the system's future performance upon changes in its parameters. The generalizability and applicability of the model is demonstrated with a real-life manufacturing environment in the food-packing industry.

2. Markov Chain Representation of Serial Production Systems

Multistage production systems have been modeled as networks with nodes representing production stages and Markovian branching indicating the movement of the product in the system. Due to the complexity of multistage production systems, simulation techniques have been most commonly used in examining such systems. Deuermeyer et al. [7] describe the above approach for semiconductor fabrication facilities. Other Markov chain representations of serial production systems include the work by Hendricks [12], and Denardo and Tang [5]. Koulamas [17] considered a single stage production system where demand, production times, machine breakdown, and repair times were stochastic. Koulamas used a Markovian approach where the states of the process characterized the status of the production state (busy, idle, undergoing repair) and its buffer (number of units in the buffer). Using the single stage system as a building block, he formed serial production systems and extracted, with the use of simulation, the probability that the last production stage falls in one of the above states. Li [21] modeled a serial production line with unreliable machines and limited repair as a continuous time Markov process with state space describing the status of the production stages as up, down-in-repair, or down-waiting for repair. The objective was to propose rules for repairing the broken machines.

In our modeling framework, serial production systems with rework are modeled as first-order Markov chains. The states of the Markov chain represent the production stages and the absorbing states, associated with each state, represent the termination of the production process and scraping of the unfinished product or the completion of the unfinished product into a finished product. Value added at each production stage may involve processing, assembly, and inspections operations or combination of the three. Forward transitions between states indicate the movement of unfinished products to subsequent production stages for further processing. Backward transitions between states indicate the movement of unfinished products to preceding production stages for rework. In this formulation, we assume that no capacity constraints exist, and that the serial production system is operating at its steady state, processing a family of products requiring the same machine sequence. The fact that in serial production systems, products undergo processing or assembly in a predefined sequence of production stages, it is natural to consider Markov chains that allow only for transitions between adjacent states.

2.1 Derivation of Absorption Probabilities

The Markov chain of the serial production system is a 2n+1 state Markov chain with transient states $\{1,2,\ldots,n\}$ and absorbing states $\{Ä_1,Ä_2,...,Ä_n,F\}$. The transient states represent the production stages of the serial production system. The absorbing states $Ä_j$, j=1,…, n represent the scraping of an unfinished product at production stage j due to defects that cannot be reworked. The absorbing state F represents the completion of the manufacturing process, implying that a product has passed through all production stages and ended up as a finished product.

Define the transition probabilities as:

$$\begin{array}{ll} P(X_{k+1} = j\text{-}1 \mid X_k=j) = q_j, & 2 \leq j \leq n, \\ P(X_{k+1} = j+1 \mid X_k=j) = p_j, & 1 \leq j \leq n\text{-}1, \\ P(X_{k+1} = Ä_j \mid X_k=j) = 1 - p_j - q_j, & 1 \leq j \leq n, \\ P(X_{k+1} = F \mid X_k=n) = p_n. & \end{array}$$

For the above Markov chain follows that $q_1 = 0$ and that $p_j \neq 0$, $\forall j$. Nevertheless, if $q_j = 0$ for some $j \in \{2,3,...,n\}$, then this shows that no backward transition is allowed at that production stage, implying that a defective unfinished product detected at stage j will not be send to the previous production stage for rework. That is, the error is irrecoverable and the product must be scraped. In this formulation, we assume that p_j for a product that has undergone rework is the same for a product that has not. In a similar way, we assume that q_j for a product that has already undergone rework is the same for a product that will undergo rework for the first time. These assumptions imply that the probability of an error occurring at a production stage may be considered constant. This stems from the fact that a machine performing an operation on a product, does not distinguish between products that have already undergone rework and products that have not.

Note that the above Markov chain formulation of serial production systems, is general enough to describe any serial production system. This is accomplished by setting the appropriate transition probabilities to zero. For example, consider the serial production system where every production stage is followed by an inspection stage and rework is performed in the production stage preceding the inspection stage at which the error is detected. By setting $q_k = 0$, $k = \{3,5,7,\ldots\}$, we describe the above serial system.

Eventually, X_k becomes absorbed in the set $\{Ä_1,...,Ä_n,F\}$. Call the limiting state $X_\propto$, and let:

$$S_{i,Äj} = P(X_\propto = Ä_j \mid X_0 = i) \quad \text{and} \quad S_{i,F} = P(X_\propto = F \mid X_0 = i)$$

denote the probabilities of absorption in state $Ä_j$ and F, respectively, starting from state i. From a practical point of view though, the quantities that are of interest are $S_{1,Äj} = P(X_\propto = Ä_j \mid X_0 = 1)$ and $S_{1,F} = P(X_\propto = F \mid X_0 = 1)$. $S_{1,Äj}$ represents the probability that a product entering the serial production system will be scraped at stage j. Equivalently, $S_{i,Äj}$ represents the percentage of the products that start from the first production stage and end up as scrap in production stage j. $S_{1,F}$ represents the yield of the serial production system at its steady state, that is, the percentage of products that start from the first production stage and end up as finished products.

In the propositions that follow we set empty products equal to 1. Proofs can be obtained from the author.

Proposition 1: Let

$$r_n = r_{n-1} = 1,\ r_{k-2} = r_{k-1} - p_{k-1} q_k r_k, \text{ for } 2 \leq k \leq n, \tag{1}$$

$$t_1 = t_2 = 1,\ t_{k+2} = t_{k+1} - q_{k+1} p_k t_k, \text{ for } 1 \leq k \leq n\text{-}1. \tag{2}$$

Then

$$S_{j,Äj} = \frac{1 - p_j - q_j}{1 - q_j\, p_{j-1}(t_{j-1} / t_j) - p_j\, q_{j+1}(r_{j+1} / r_j)}, \quad 1 \leq j < n, \tag{3}$$

$$S_{n,Än} = \frac{1 - p_n - q_n}{1 - q_n\, p_{n-1}(t_{n-1} / t_n)}, \tag{4}$$

$$S_{1,Äj} = \Big(\underset{i=1}{\overset{j-1}{Ð}} p_i\Big)(S_{j,Äj} / t_j), \quad 1 \leq j \leq n \tag{5}$$

$$S_{1,F} = [p_n / (1 - p_n - q_n)]\Big(\underset{i=1}{\overset{n-1}{Ð}} p_i\Big)(S_{n,Än} / t_n) \tag{6}$$

2.2 Expected Costs and Quantities for Products Ending in Any of the Absosrbing States

The Markov chain modeling framework can also be used to compute various expected costs and quantities associated with the serial production system. Such expected costs and quantities include the expected number of production stages visited before a product is scraped or completed, the expected time until a product is scraped or completed, the expected cost of a product whether scraped or completed, the expected number of states visited for rework by a product whether scraped or completed, and the expected time and cost allocated for rework for products scraped or completed. These values represent measures of performance of the serial production system.

To compute the first three of the above costs and quantities, we associate each state k with a number e_k which can be viewed as the “weight” or “cost” of visiting state k. For example, setting $e_k=1$ gives a cost of 1 every time a state is entered. Therefore, summing over all states yields the total number of times states have been entered, or equivalently, the total number of production stages visited. Similarly, e_k may represent the average cost accumulated while in state (production stage) k or the average time required for the production operation in stage k. Average or fixed values for production costs and processing times have also been used in formulations of serial production systems by Lee and Yano [20] and Akella et al.[2].

The proposition that follows, presents the formulae for the first three costs and quantities (expected number of production stages visited before absorption, expected time until absorption, and expected cost of a product absorbed in any absorbing state) associated with a serial production system. Subsequently, the results of the proposition will be used to extract the other three quantities (expected number of states visited by a product for rework, expected time and cost spend on rework). Note that the expected costs and quantities described in this subsection are related to *all* products ending in absorbing states, whether finished or scraped. In the proposition that follows, we denote by C to be the total cost associated with one of these costs and quantities.

Proposition 2: Let

$$c_{n-1} = q_n,\ d_{n-1} = e_n,\ f_2 = p_1,\ g_2 = e_1 \qquad (15)$$

and

$$c_{k-1} = q_k / (1-p_k c_k),\ d_{k-1} = (e_k+p_k d_k) / (1-p_k c_k) \text{ for } 2 \le k < n, \qquad (16)$$

$$f_{k+1} = p_k / (1-q_k f_k),\ g_{k+1} = (e_k+q_k g_k) / (1-q_k f_k) \text{ for } 2 \le k < n, \qquad (17)$$

Then

$$C = (bc+d) / (1-ac)$$

where

$$a= \prod_{j=1}^{n-1} c_j,\quad b=\sum_{j=1}^{n-1} d_j \prod_{i=j+1}^{n-1} c_j,\quad c= \prod_{j=2}^{n} f_j,\quad d= \sum_{j=2}^{n} g_j \prod_{i=2}^{j-1} f_i \qquad (18)$$

For the special case where the production system does not allow rework, implying $q_k=0$, $\forall k$, the formulae in Propositions 1 and 2 simplify considerably. The proposition that follows presents these formulae.

Propositions 3: If $q_k=0$, $\forall k$, then

$$S_{1,\Delta j} = (1-p_j) \prod_{i=1}^{j-1} p_i,\ 1 \le j < n,$$

$$S_{1,\Delta n} = (1-p_n) \prod_{i=1}^{n-1} p_i, \qquad S_{1,F} = p_n \prod_{i=1}^{n-1} p_i,$$

$$C = \sum_{j=1}^{n} e_j \prod_{i=1}^{n-1} p_i. \qquad (22)$$

From Proposition 3, we observe that when $q_k=0$, the absorption probability formulae reduce to a simple multiplication of the transition probabilities. In addition, we note that when $e_k=1$, that is, when C represents the number of state visits before absorption, we can compute C alternatively by arguing that for $1 \le j < n$ we visit j states before absorption with probability $(1-p_j)\prod_{i=1}^{j-1}p_i$ and n states with probability $\prod_{i=1}^{n-1}p_i$. Therefore,
$C= \sum_{j=1}^{n-1}j(1-p_j) \prod_{i=1}^{j-1}p_i + n \prod_{i=1}^{n-1}p_i$, which is equivalent to C in Proposition 3 when $e_k=1$.

An additional observation from Propositions 1 and 3 is that the subtraction of $S_{1,F}$ in Proposition 3 from the respective value in Proposition 1 yields an important quantity for a serial production system, namely the improvement in yield attributed to the rework operation.

Given the formulae in Propositions 2 and 3 we can now extract all expected costs and quantities mentioned above for *all* products ending in any of the absorbing states. Products that end up in the absorbing states Δ_j, j= 1,…., n, of production stage j, indicate that the error detected can not be reworked, resulting in scraping the product. Furthermore, products may end up in the final absorbing state F indicating the completion of the product into a finished product.

Expected number of state visits. As stated above, by setting $e_k=1$, k=1,….,n, the cost formulae in Proposition 2 will yield the expected number of states visited by all products ending in one of the absorbing states, including F. Define this cost as $States_{all}$.

Expected time until a product ends in an absorbing state. If we allow e_k to represent the average time required for processing, assembly, inspection operations or any combination of them in state (production stage) k, the cost formulae in Proposition 2 will yield the expected time it takes a product, scraped or finished, to end up in an absorbing state. Define this cost as $Time_{all}$.

Expected cost of production with rework. Similarly, if we set e_k to represent the average cost of production at stage k, the cost formulae in Proposition 2 will yield the expected cost of all products ending in absorbing states, whether these products are scraped or finished. Define this cost as $Cost_{all}$.

Expected cost of rework. To compute the expected cost of rework for all products ending in absorbing states, simply note that from Proposition 2 we can compute the expected cost of production with rework and from Proposition 3 we can compute the expected cost of production for a system when no rework is permitted. If for a serial production system with rework we set $q_k=0$, $\forall k$, then the system degenerates to one where no rework is permitted and any defective product detected at any production stage of the production process will be scraped. Therefore, the cost of rework for a serial production system with rework is given by $C|_{q1=0}$ - $C|_{qk=0,\forall k}$, where C is given Proposition 2. Define this cost as $\text{Rework}_{\text{all}}$.
Expected number of states visited for rework. Similarly, the expected number of states visited for rework by all products, whether they end up as scrap in an absorbing state or as finished products in the absorbing state F, is given by $C|_{q1=0}$ - $C|_{qk=0,\forall k}$, when we set $e_k=1$. The function C is given in Proposition 2. Define this quantity as $\text{States(Rework)}_{\text{all}}$.
Expected time spend on rework. To compute the expected time spend on rework by all products ending up in any of the absorbing states including F, we simply obtain $C|_{q1=0}$ - $C|_{qk=0,\forall k}$, when e_k represents the average time spend while in state (production stage) k. The function C is given in Proposition 2. Define this quantity as $\text{Time(Rework)}_{\text{all}}$.

2.3 Expected Costs and Quantities Associated with Finished Products

The expected costs and quantities computed above are associated with all products ending in any of the absorbing states, whether scraped or finished products. These costs are extremely useful to management since they can be considered as measures of effectiveness of the serial production system. However, management's attention usually focuses on costs and quantities associated with the finished products.

To compute the related costs and quantities for the finished products only, assume that the serial production system has processed N products. This number includes all products ending in all absorbing states whether they are scraped products or finished goods. In other words, N represents the number of products that started the first production stage and either end up as scraped products in one of the absorbing states $Ä_j$, j=1,…., n, or as finished goods in state F. Given the above, the expected total number of states visited for the production of these N products is given by N x $\text{States}_{\text{all}}$. Since $S_{1,F}$ represents the yield of the serial production system, $S_{1,F}$ x N represents the expected number of finished products in the case where N products end up in any of the absorbing states. If we define $\text{States}_{\text{finished}}$ as the number of states visited, that a finished product must be "burdened" for its production then,

$$\text{States}_{\text{finished}} = (N \text{ x } \text{States}_{\text{all}}) / (N \text{ x } S_{1,F}) = \text{States}_{\text{all}} / S_{1,F}$$

In other words, for every finished product, $\text{States}_{\text{finished}}$ must be visited. Following a similar argument as the one above, the expected production time that a finished product is "burdened" with for its production is given by $\text{Time}_{\text{finished}} = (N \text{ x } \text{Time}_{\text{all}}) / (N \text{ x } S_{1,F}) = \text{Time}_{\text{all}} / S_{1,F}$. In general, if we divide the costs computed for all the products ending in absorbing states by the yield of the serial production system $S_{1,F}$, we obtain the respective costs for the finished products only. As such, $\text{Cost}_{\text{finished}} = \text{Cost}_{\text{all}} / S_{1,F}$ represents he actual cost of a finished product. We can also compute the above cost by summing up the cost of production, the cost of rework, and all the value added to products that have been scraped and divide this sum by the number of finished products.

The quantities $\text{States(Rework)}_{\text{finished}} = \text{States(Rework)}_{\text{all}} / S_{1,F}$, $\text{Time(Rework)}_{\text{finished}} = \text{Time(Rework)}_{\text{all}} / S_{1,F}$, and $\text{Cost(Rework)}_{\text{finished}} = \text{Cost(Rework)}_{\text{all}} / S_{1,F}$ represent the expected number of states visited for rework, time spend on rework, and the cost of rework, respectively, that a finished product is "burdened" with. An important quantity that can be estimated from the Markov chain formulation of a serial production system, is the expected cost of scrap that each finished product is "burdened" with. Define $\text{Scrap}_{\text{finished}}$ as the cost that a finished product is "burdened" with from products that have been scraped during the production process. This cost is given by

$$\text{Cost}_{\text{finished}} - \sum_{k=1}^{n} e_k - \text{Cost(Rework)}_{\text{finished}} < \text{Scrap}_{\text{finished}} < \text{Cost}_{\text{finished}} - \sum_{k=1}^{n} e_k,$$

where e_k represents the average cost of production at production stage k. The upper bound of the cost of scrap equals to the true expected cost of production of a finished product minus the cost of rework for a finished product as derived above. In other words, the upper bound assumes that all products reworked are scraped. Since though, not all products reworked end up as scrap, the theoretical minimum for the cost of scrap, and thus the lower bound, is reflected in the case where no product that has been reworked ends up as scrap.

3. Application of the Modeling Framework

The applicability of the model presented in this paper is demonstrated in a real life manufacturing environment. The company in question performs honey-packing operations. The production process consists of three stages that are arranged serially. During the first stage, glass jars received from the glass manufacturers are removed from pallets and are placed on a conveyor belt. During the unloading process, glass jars may break resulting in the scraping of the whole jar. Breakages occur more frequently in the case where the plant packs small jars with delicate handles. During he second stage, the glass jars are moved by the conveyor belt to pneumatic filling machine that releases a prespecified quantity of warm honey in the jars. Problems in this stage may be caused from misplaced jars, which lead to spillovers, by the filling machine. In such cases, quantities of honey are lost and several jars need to be reworked, i.e., cleaned in order to be reused. In addition, rework appears in the form of returning quantities of honey to the storage tank from the jars that are to be cleaned. Finally, during the third and final stage, lids are placed on the jars emerging from the filling machine (second stage). In addition, a label is attached to the jars by a labeling machine.

Eventually, jars are manually packed in carton boxes. Defects in this stage result from mis placed labels and/or lids, and from breakages during the manual carton-packing process. Rework at this stage includes the removal of labels or lids and the proper labeling and capping.

The transition probabilities of the three stages were supplied by the firm's management based on historical data. The values of the transition probabilities are equal to p_1=0.96, p_2=0.97, p_3=0.96, and q_1=0, q_2=0.02, q_3=0.02. The average times and costs needed at the three production stages are given by the vectors (1,2,2) and (2,3,1), respectively. The "low" forward transition probabilities of the three stages were attributed by management to human error resulting from the manual operations and the "aging" machinery.

Table 1 presents the absorption probabilities of each production stage. Of particular interest is the value of $S_{1,F}$=0.92984 which represents the probability that a product starting from the first production stage will end up as a finished product. As such, this value represents the yield of the serial production system. The table also includes the various expected costs and quantities discussed earlier for all products ending in absorbing states and also the respective costs and quantities for the finished products only. Consider for example the costs and quantities associated with all products ending in absorbing states. The expected number of states visited before absorption is equal to 2.98710. If the production stages of the serial system were free of quality problems, that is p_k=1, $\forall$k, then the number of states visited would have been three. The expected number of states visited before absorption is less than three since in the serial production system presented in this paper, products get absorbed before visiting all states. As such, the expected number of states visited can be viewed as a performance measure of the serial system.

From Table 1 we also see that the expected time needed for a product to be absorbed is equal to 4.95423. Since the yield of the three production stages is less than one, this implies that the expected time required for a product to be absorbed is less than the time need for the completion of a finished product (in the case where no quality problems are present). As mentioned above, this occurs when p_n=1, $\forall$k. The closer the expected time until a product is absorbed to five (the sum of times at each production stage) the better the performance of the serial production system. Similarly, the closer the expected cost of production with rework to six (the sum of costs at each production stage) the better the performance of the system. Furthermore, the closer the expected number of states visited for rework, expected time spend on rework, and expected cost of rework are to zero, the better the performance of the system.

Table 1 also presents the true costs associated with the finished products. For example, consider the expected number of states visited which is equal to 3.21248. This implies that for every finished product that is completed, 3.21248 states have to be visited. In the case where no quality problems are present, a product is required to visit three production stages for its completion. The difference of 0.21248 can be attributed to the inefficiency of the system due to quality problems and rework. Similarly, every finished product produced is "burdened" with 5.32803 units of production time. A product undergoing production with no quality problems and thus no rework, would require five units of time. The difference represents the excess production time that a finished product is burdened with due to quality problems. A more important value is the true expected cost of a finished product which is equal to 6.45717. A product that undergoes production without any quality problems, would require a cost of six. Therefore, the difference between the two values represents the excess cost due to the scraped material and rework performed. When improving the serial system, the expected cost of finished products should be as close as possible to six. Similarly, the expected cost of rework should be as close as possible to zero.

The analysis in this paper can also assist us to determine a range for the cost of scrap. Since the expected cost of a finished product is 6.45717 and the cost without any quality problems is six, the difference of 0.45717 includes: (1) the cost of rework whose value, from Table 1, is equal to 0.20752 and (2) the per finished product expected cost of scrap. Note that some of the products reworked end up as scrap. This implies that 0.45717 is the theoretical maximum cost of scrap. The minimum theoretical cost of scrap is given by 0.45717 - 0.20752 = 0.24965, which represents the case where no product that has been reworked is scraped. Improvements in the quality of the system would translate into a lower minimum and maximum cost of scrap.

Absorbing State	p_k	q_k	Absorption Probability
Ä$_1$	0.96	0.00	0.04080
Ä$_2$	0.97	0.02	0.00999
Ä$_3$	0.96	0.02	0.01937
F			0.92984

Costs (For all products ending in absorbing states)	e_k	
Expected number of state visits	(1,1,1)	2.98710
Expected time until a product is completed	(1,2,2)	4.95423
Expected cost of production with rework	(2,3,1)	6.00416
Expected number of state visited for rework		0.09590
Expected time spend on rework		0.17183
Expected cost of rework		0.19296

Costs (For finished products)	
Expected number of state visits	3.21248
Expected time until a product is completed	5.32803
Expected cost of production with rework	6.45717
Expected cost of rework	0.20752
Expected cost of scrap	0.24965 < cost of scrap < 0.45717

Table 1: Absorption probabilities and expected costs and quantities from a food-packing serial production system.

4. Conclusion

Serial production systems with rework can be represented as a first-order Markov chain. Using this modeling framework, we derive formulae for the final absorption probabilities which represent: (1) the probabilities that a product gets scraped at a certain production stage and (2) the probability that a product ends up as a finished product, which is equivalent to the yield of the system. In addition, we derive various expected costs and quantities for the serial production system. The model can be used to assess changes on the system's yield, costs, and quantities caused by (1) changes in the transition probabilities of a product moving to the next or previous production stage and (2) changes in various parameters associated with each production stage, such as the average time spend in the stage or the cost while in this stage. This provides insights as to where management should focus its efforts when improving the system. It allows through a sensitivity analysis to see the system's future performance upon changes in its parameters. The applicability of the framework is demonstrated with a real-life example from the food-packing industry.

Directions of future research include the modeling of a serial production system which adopts batch processing instead of continuous processing. In addition, as in Lee and Yano [20], the analysis can be extended to include distributions of yield at each production stage. Similarly, we may consider the above analysis for serial systems where processing times at each production stage follow a distribution. As evident from the application of the modeling framework, improvements in the yield of the production stages of a serial system are more cost effective when performed first at the final stages of production. This result can be further pursued in order to obtain an analytical proof which is generalizable for any multistage production system.

References

[1] Agnihothri, S.R., and Kenett, R.S., ``The impact of Defects on a Process with Rework", *European Journal of Operational Research*, **80**, 308-327 (1995).

[2] Akella, R., Rajagopalan, S., and Singh, M.R., ``Part Dispatch in Random Yield Multistage Flexible Test Systems for Printed Circuit Boards," *Operations Research*, **40**, 776-789 (1992).

[3] Ballou, D.P., and Pazer, H.L., ``The Impact of Inspector, Fallibility on the Inspection Policy in Serial Production Systems," *Management Science*, **28**, 387-399 (1982).

[4] Conway, R., Maxwell, W., McClain, J.O., and Thomas, L.J.,``The Role of Work-In-Process Inventory in Serial Production Lines," *Operations Research*, **36**, 229-241 (1988).

[5] Denardo, E.V., and Tang, C.S., ``Linear Control of a Markov Production System," *Operations Research*, **40**, 259-278 (1992).

[6] Denardo, E.V., and Lee, T.Y.S., ``Managing Uncertainty in a Serial Production Line," *Operations Research*, **44**, 382-392 (1996).

[7] Deuermeyer, B.L., Curry, G.L., and Feldman, M., ``An Automatic Modelling Approach to the Strategic Analysis of Semiconductor Fabrication Facilities," *Production and Operations Management*, **2**, 195-220 (1993).

[8] Fisher, M.L, and Ittner, C.D., ``The Impact of Product Variety on Automobile Assembly Operations: Empirical Evidence and Simulation Analysis," *Management Science*, **45**, 771-786 (1999).

[9] Gong, L., de Kok, T., and Ding, J., ``Optimal Lead Times Planning in a Serial Production System," *Management Science*, **40**, 629-632 (1994).

[10] Grosfeld-Nir, A., and Robinson, L.W., ``Production to Order on a Two Machine Line with Random Yields and Rigid Demand," *European journal of Operational Research*, **80**, 264-276 (1995).

[11] Harrison, T.P., and Lewis, H.S., ``Lot Sizing in Serial Assembly Systems with Multiple Constrained Resources," *Management Science*, **42**, 19-36 (1996).

[12] Hendricks, K.B., ``The Output Process of Serial Production Lines of Exponential Machines with Finite Buffers," *Operations Research*, **40**, 1139-1147 (1992).

[13] Hendricks, K.B., and McClain, J.O., ``The Output Process of Serial Production Lines of General Machines with Finite Buffers," *Management Science*, **39**, 1194-1201 (1993).

[14] Hwang, J., and Singh, M.R., ``Optimal Production Policies for Multi-Stage Systems with Setup Costs and Uncertain Capabilities," *Management Science*, **44**, 1279-1294 (1998).

[15] Karimi, I.A., ``Optimal Cycle Times in Multistage Serial Systems with Set-Up and Inventory Costs," *Management Science*, **38**, 1467-1481 (1992).

[16] Kimura, O., and Terada, H., ``Design and Analysis of Pull Systems, a Method of Multistage Production Control," *International Journal of Production Research*, **19**, 241-253, (1981).

[17] Koulamas, C.P.\ ``Single-Stage and Serial Production Line Systems with Unreliable Machines and General Probability Distribution," *Decision Sciences*, **24**, 343-369 (1993).

[18] Lee, H.L., ``Lot Sizing to reduce Capacity Utilization in a Production Process with Defective Items, Process Corrections, and Rework," *Management Science*, **38**, 1314-1328 (1992).

[19] Lee, H.L., ``Input Control for Serial Production Lines Consisting of Processing and Assembly Operations with Random Yields," *Operations Research*}, **44**, 464-468 (1996).

[20] Lee, H.L., and Yano, C.A., ``Production Control in Multistage Systems with Variable Losses," *Operations Research*, **36**, 269-278 (1988).

[21] Li, K.-F., ``Serial Production Lines with Unreliable Machines and Limited Repair," *Naval Research Logistics*}, **34**, 101-108 (1987).

[22] Mohan, R.P., and Ritzman, L.P., ``Planned Lead Times in Multistage Systems," *Decision Sciences*, **29**, 163-191 (1998).

[23] Rebello, R., Agnetis, A., and Mirchandani, P.B., ``Specialized Inspection Problems in Serial Production Systems," *European Journal of Operational Research*, **80**, 277-296 (1995).

[24] Sarkar, D.S., and Zangwill, W.I., ``File and Work Transfers in Cyclic Queue Systems," *Management Science*, **38**, 1510-1523 (1992).

[25] So, K.C., and Tang, C.S., ``Optimal Operating Policy for a Bottleneck with Random Rework," *Management Science*}, **41**, 620-636 (1995).

[26] Tang, C.S., ``The Impact of Uncertainty on a Production Line," *Management Science*, **36**, 1518-1531 (1990).

[27] Wein, A.S., ``Random Yield, Rework and Scrap in a Multistage Batch Manufacturing Environment," *Operations Research*, **40**, 551-463 (1992).

[28] Yano, C.A., ``Setting Planned Leadtimes in Serial Production Systems with Tardiness Costs," *Management Science*, **33**, 95-106 (1987).

[29] Yano, C.A., and Lee, H.L., ``Lot Sizing with Random Yields: A Review," *Operations Research*, **43**, 311-334 (1995).