# Group Maintenance Models for Unreliable Service Systems - Maintenance Decision Depending on Number of Customers

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#### Abstract

In this research we are going to develop group maintenance policies for a service/production system by considering the number of customers dynamically in the queue. It means that we generate the related group maintenance models by allowing the maintenance decision to depend on the number of customers in the system at any given time.

Group maintenance models can be applied in many fields of business and military including production process of manufacturing factory, service process of service-providing company, maintenance process of military weapon system, etc.. The main idea of group maintenance is to delay the initiation of the maintenance process until more servers have failed, so that the fixed repair cost is distributed over more servers.

The typical approach in the analysis of group replacement policies is to assume that the operating machines produce output at a constant rate. It is maybe realistic for automated high-volume production processes of manufacturing company, but definitely not the case for service-providing company, such as bank, telecommunication business, internet-related service provider, etc. Therefore, it is more reasonable to assume such service systems where the customers/jobs arrive according to a random stream.

In this research we consider a production/service system with multiple independent servers operating in parallel and a single Markovian queue. The servers are unreliable with identically exponentially distributed failure times and the repair time is also assumed to follow exponential distribution. The repair cost consists of a fixed cost associated with starting the repair process and a variable cost proportional to the number of repaired servers. Customers arrive in accordance with a Poisson process, and the service time for each customer follows an exponential distribution. Then we can obtain more precise customer holding and loss cost than the traditional group replacement models in which generally assume constant production job arrivals. Finally, we try to develop a specific class of group maintenance policy: A customer-dependent group maintenance model, where the decision on initiating repairs depends on the number of customers present in the system. We formulate this model as a continuous time Markov decision process, and show that the optimal group maintenance policy has a threshold structure.

**Keywords:** Group maintenance policy; Group replacement policy; Customer-dependent group maintenance policy; customer holding cost; customer loss cost; repair cost; opening/setting up

#### **1. Introduction**

A large amount of research has been devoted to finding optimal replacement policies of the group replacement models. According to our observation, there are three main types of group replacement policies, which have been studied in most of the literature. The first one is Tage group replacement policy. The main idea for this policy is that no failed machine is repaired until a scheduled time T. Then all failed machines in the system are repaired simultaneously. In Barlow, Proschan, and Hunter, it is shown that the optimal scheduled time for preventive maintenance is nonrandom and there exists a unique optimal policy if the distribution of time to failure has an increasing failure rate. A detailed analysis for determining optimal T is presented in Okumo to & Elsayed. For the case of exponential distribution, a closed form expression for T\* is developed. For general underlying failure distribution, bounds for T\* are derived. The second one is M-failure replacement policy. The main idea for this policy is that we do not repair any failed machine until m failed machines have occurred. Then all failed machines in the system are fixed at the same time. Assaf & Shanthikumar has considered two models with exponential failure times with parameter  $\lambda$  and a more general replacement policy f(m,n): when the number of failed machines reaches m, n machines are repaired. They show that the optimal repair policy is either not to repair any failed machine or to repair with an f(m,m)-type policy. This is effectively a m-failure group replacement policy. By extending Assaf & Shanthikumar's repair and replacement model, optimal m failure policies with nonnegative random repair time are discussed in Wilson & Benmerzouga. The last one is (m,T) group replacement policy. The main idea for this policy is that we do not repair any failed machine until

a scheduled time T or upon m failed machines whichever comes first. Then all failed machines in the system are repaired at

the same time. Nakagawa has considered the optimal number  $m^*$  to minimize the mean cost rate when the scheduled replacement time T is specified. Ritchken & Wilson have considered a generalization of the combined (m,T) group replacement model which requires inspection at either the scheduled time T or the time when exactly m machines have failed, whichever comes first. At an inspection, all failed machines are replaced with new units while operating machines are serviced so that they become as good as new.

Also, the analysis of the group replacement model is somewhat related to or based on that of a single machine with minimal repair and replacement options. A comprehensive discussion on different maintenance policies for this kind of model is included in Beichelt. The mathematical background for analyzing maintenance policies with minimal repair is presented and Standard maintenance policies are summarized.

The typical approach in the analysis of group replacement policies is to assume that the operating machines produce output at a constant rate, that is, there is a continuous input flow of production jobs into the system. This means that when a failed machine is left un-repaired, there is a production loss cost incurred at a constant rate. Although the assumption of continuous inflow is appropriate for high-volume production processes, it may be realistic for systems where the jobs arrive according to a random arrival stream. In this case, the production loss cost rate is not constant, but instead it depends on the number of jobs waiting for service at any given time. It means that we can obtain more precise production holding cost to respond the random situation of real world than the traditional group replacement models in which we assume constant production job arrivals.

The group replacement problems we described in this paper also fall in two categories of queuing system with unreliable servers. In the first category, the matrix-geometric method for steady state analysis of a certain class of continuous time Markov processes is usually applied. The earliest results on matrix-geometric solutions are contained in the paper of Evans and the Ph.D. thesis of Wallace for block-Jacobi generators of continuous-parameter Markov processes of the GI/M/1 type, called quasi birth and death (QBD) processes. In Neuts & Lucantoni and Neuts, a M/M/N queueing system is analyzed and modeled as a continuous time Markov chain with state (x,w), where x denotes the number of customers in the system, and w the number of operating servers. The steady state probability vector is shown to be of matrix-geometric type. The average system length and waiting time distribution are also calculated. In the second category, the semi-Markov decision process is usually applied to solve this kind of repair problems. Federgruen & So (1989) considers a single server queueing system with Poisson arrivals and general service times. While the server is working, the breakdowns are subject to a Poisson process. When the server is failed, it needs to decide to repair the server immediately or delay the repair of failed server until the number of customers in the system exceeds certain threshold of an optimal stationary policy. By extending the previous results, a modified model is developed by Federgruen & So (1990) to repair the failed server immediately by initiating one of two available repair operations. They have proved a weaker result: There exists an optimal stationary policy which applies the faster repair if and only if the number of customers in the system exceeds a certain threshold.

In this research we consider the general class of group replacement policies which allow the maintenance decision to depend on the number of customers in the system as well as the number of operating servers at any given time. These policies are expected to be more cost-efficient than the standard group replacement policies that depend on the number of servers only. The reason is that if at a given instant several servers have failed but the system is not overcrowded, it may be beneficial to postpone the group replacement in order to allow for more servers to fail before the replacement is started, so that the fixed repair cost is distributed to more servers. At the same time the customer delay costs are not too high if the number of customers present is not too large. Because the replacement action is now dynamic, the problem of finding a maintenance policy to minimize any cost can be formulated as a continuous-time Markov decision process.

## 2. Problem Description and Model Formulation

In this research we consider a production system with N independently operating servers and a single queue. Customers arrive in accordance with a Poisson process with rate ë, the service time for each customer follows exponential distribution with rate i. The servers are unreliable with identically exponentially distributed failure times and the failure rate of each server is f. In the repair process the repair time follows exponential distribution. When the repair process is initiated, repairs are performed by a crew of c repairmen. The repair rate for fixing one machine by using one repairman is r. We assume that the repair crew devotes their effort proportionately to all machines being

repaired at any given time. Therefore, the instantaneous repair rate is equal to  $r_w = \frac{rc}{N-w}$ , when w machines are

operational and N-w under repair. Within the period of maintenance, other transitions such as customer arrival, customer service and machine failure are also allowed to happen. Each time the repair process is initiated, there exist a fixed cost S and a variable cost  $r_c$  per repaired server. In addition a holding cost h is incurred per unit of time for every customer present in the system. The existence of the fixed replacement cost makes it desirable to postpone replacement until a substantial number of servers have failed. On the other hand, the delay costs increase when failed servers are left un-repaired. The objective is to find a dynamic policy to determine when to perform the group replacement, in order to minimize the infinite horizon discounted replacement and holding cost per unit of time.

The system described above can be formulated in terms of a continuous-time Markov decision process as follows:

1. The state is denoted by (x,w), x = 0, and 0 = w = N, where x denotes the number of customers in the system, and w the number of working servers in the system. 2. The action space in state (x,w) is A={1,2}, where a=1 means the production process is continued and a=2 means that the group maintenance of failed servers is performed.

3. Transition mechanism:

- (1) When a=1 (continue)
  - $(x,w) \rightarrow (x+1,w)$  with exponential transition rate  $\ddot{e}$ .
  - $(x,w) \rightarrow (x,w-1)$  with exponential transition rate wf.
  - $(x,w) \rightarrow (x-1,w)$  with exponential transition rate  $\min(x,w)$ .
- (2) When a=2 (do the group maintenance)

 $(x,w) \rightarrow (x+1,w)$  with exponential transition rate  $\ddot{e}$ .

 $(x,w) \rightarrow (x,w-1)$  with exponential transition rate wf.

 $(x,w) \rightarrow (x-1,w)$  with exponential transition rate  $i \min(x,w)$ 

$$(x,w) \rightarrow (x,N)$$
 with exponential transition rate  $\frac{cr}{N-w}$ 

An equivalent way to describe the transition process is the following:

(1) when a=1, the process remains in state (x,w) for an exponentially distributed time with rate  $\lambda(x, w) = \ddot{e} + wf + i \min(x,w)$ . After a transition occurs, it will move to one of states (x+1,w), (x,w-1), (x-1,w) with respective probabilities :

$$P_{(x,w)(x+1,w)}(a=1) = \frac{\lambda}{\lambda(x,w)}, \ P_{(x,w)(x,w-1)}(a=1) = \frac{wf}{\lambda(x,w)}, \ P_{(x,w)(x-1,w)}(a=1) = \frac{\mu\min(x,w)}{\lambda(x,w)},$$

(2) when a=2, the process remains in state (x,w) for an exponentially distributed time with rate  $\ddot{e}+wf+imin(x,w)+\frac{cr}{N-w}$ . After a transition occurs, it will move to one of states (x+1,w), (x,w-1), (x-1,w), (x,N) with

respective probabilities:

$$\begin{split} P_{(x,w)(x+1,w)}(a = 2) &= \frac{\lambda}{\lambda(x,w)} \qquad , \qquad P_{(x,w)(x,w-1)}(a = 2) = \frac{wf}{\lambda(x,w)} \qquad , \qquad P_{(x,w)(x-1,w)}(a = 2) = \frac{\mu\min(x,w)}{\lambda(x,w)} \qquad , \\ P_{(x,w)(x,N)}(a = 2) &= \frac{cr}{N-w} \\ \lambda(x,w) \qquad . \end{split}$$

Conditional on the event that the next state is (x, w), the time until the transition from(x, w) to (x, w) is a random variable with exponential distribution  $F_{(x,w)(x,w)}$  ( $|a\rangle$ :

$$\begin{split} F_{(x,w)(x+1,w)}(t \mid a = 1) &= F_{(x,w)(x,w-1)}(t \mid a = 1) = F_{(x,w)(x-1,w)}(t \mid a = 1) = 1 - e^{-(\lambda + wf + \mu \min(x,w))t}, \\ F_{(x,w)(x+1,w)}(t \mid a = 2) &= F_{(x,w)(x,w-1)}(t \mid a = 2) = F_{(x,w)(x-1,w)}(t \mid a = 2) = F_{(x,w)(x,N)}(t \mid a = 2) \\ &= 1 - e^{-(\lambda + wf + \mu \min(x,w))t + cr} N_{-w} \end{split}$$

#### 4. Cost structure:

If action a is chosen when in state (x, w), then an immediate costC((x, w), a) is incurred and, in addition, a cost rate c((x, w), a) is imposed until the next transition occurs:

C((x,w),a=1)=0,  $C((x, w),a=2) = S + (N - w)_{r_c}, c((x,w),a=1) = hx$ , c((x,w),a=2) = hx,

where S is the fixed cost for starting repairs,  $r_c$  is the variable repair cost for each machine replaced, and h is the holding cost per customer and per unit time. If a transition occurs after t units, then the total cost incurred is given by C((x,w),a) + t c((x,w),a).

# 3. Discounted Cost Criterion

Assume that costs are continuously discounted with a discount rate  $\alpha = 0$ , and consider minimizing the expected total discounted cost V(x,w). Using standard results from the theory of continuous time MDP (Puterman, 1994), we can transform this model into a discrete time MDP with decision epochs at transitions. The details of the transformation are presented as follows:

1. The one step cost  $\overline{C_{\alpha}}((x,w),a)$  equivalent of the discrete time process is

$$\overline{C_{\alpha}}((x, w), a) = C((x, w), a) + \sum_{(x, w)} P_{(x, w)(x, w)}(a) \int_{0}^{1} e^{-\alpha s} c((x, w), a) ds dF_{(x, w)(x, w)}(t \mid a).$$

In particular, when a=1 (continue), then

$$\overline{C_{\alpha}}((x, w), 1) = 0 + \int_{0}^{t} e^{-\alpha s} ds hx \lambda(x, w) e^{-\lambda(x, w) t} dt = \frac{hx}{\alpha + \lambda(x, w)}$$
(3.1)

Also, when a=2 (do the group maintenance),

or

According to the above transition mechanism, note that the probability of finishing the maintenance is

$$\frac{\frac{cr}{N-w}}{\alpha+\lambda(x,w)+\frac{cr}{N-w}}$$
. Therefore, the expected repair cost C((x,w),a=2)= $\frac{\frac{cr}{N-w}}{\alpha+\lambda(x,w)+\frac{cr}{N-w}}$ [S+(N-w)r<sub>c</sub>], which

reflects the property that the maintenance cost is incurred only if the maintenance is completed before other transitions happened. Now we have one step cost as follows:

$$\overline{C_{\alpha}}((x, w), 2) = \frac{\overline{N-w}}{\alpha + \lambda(x, w) + \frac{cr}{N-w}} [S + (N - w)_{r_{c}}] + \int_{0}^{t} e^{-\alpha s} hxdsd(1 - e^{-[\lambda(x, w) + \frac{cr}{N-w}]t})$$

$$= \frac{\overline{N-w}}{\alpha + \lambda(x, w) + \frac{cr}{N-w}} [S + (N - w)_{r_{c}}] + \frac{hx}{\alpha + \lambda(x, w) + \frac{cr}{N-w}}$$
(3.2)

2. The future discounted cost is

$$\overline{V}((x, w), a) = \prod_{(x,w)} P_{(x,w)(x,w)}(a) e^{-\alpha t} V_{\alpha}((x, w)) dF_{(x,w)(x,w)}(t \mid a):$$

In particular, when a=1, after some intermediate simple integration:

$$\begin{split} \overline{V}((\mathbf{x},\mathbf{w}),1) &= \sum_{(\mathbf{x},\mathbf{w})} P_{(\mathbf{x},\mathbf{w})(\mathbf{x},\mathbf{w})} (1) e^{-\alpha t} V_{\alpha}((\mathbf{x},\mathbf{w}) ) \lambda(\mathbf{x},\mathbf{w}) e^{-\lambda(\mathbf{x},\mathbf{w})t} dt \\ &= \sum_{(\mathbf{x},\mathbf{w})} P_{(\mathbf{x},\mathbf{w})(\mathbf{x},\mathbf{w})} (1) V_{\alpha}((\mathbf{x},\mathbf{w}) ) (\frac{\lambda(\mathbf{x},\mathbf{w})}{\alpha + \lambda(\mathbf{x},\mathbf{w})}) \\ &= \left[\frac{\lambda}{\lambda(\mathbf{x},\mathbf{w})} V_{\alpha}(\mathbf{x}+1,\mathbf{w}) + \frac{wf}{\lambda(\mathbf{x},\mathbf{w})} V_{\alpha}(\mathbf{x},\mathbf{w}-1) + \frac{\mu\min(\mathbf{x},\mathbf{w})}{\lambda(\mathbf{x},\mathbf{w})} V_{\alpha}(\mathbf{x}-1,\mathbf{w})\right] \frac{\lambda(\mathbf{x},\mathbf{w})}{\alpha + \lambda(\mathbf{x},\mathbf{w})} \\ &= \frac{\lambda}{\alpha + \lambda(\mathbf{x},\mathbf{w})} V_{\alpha}(\mathbf{x}+1,\mathbf{w}) + \frac{wf}{\alpha + \lambda(\mathbf{x},\mathbf{w})} V_{\alpha}(\mathbf{x},\mathbf{w}-1) + \frac{\mu\min(\mathbf{x},\mathbf{w})}{\alpha + \lambda(\mathbf{x},\mathbf{w})} V_{\alpha}(\mathbf{x}-1,\mathbf{w}) \quad (3.3) \end{split}$$

Similarly, when a=2,

$$\overline{V}((x, w), 2) = \frac{\lambda}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} V_{\alpha}(x + 1, w) + \frac{wf}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} V_{\alpha}(x, w - 1)$$

$$+\frac{\mu\min(x, w)}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} V_{\alpha}(x - 1, w) + \frac{\frac{cr}{N - w}}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} V_{\alpha}(x, N)$$
(3.4)

3. Using the above results, the optimality equations for the discrete time version of problem can be expressed as  $V_{\alpha}(x, w) = Min \{V_{\alpha}^{1}(x, w), V_{\alpha}^{2}(x, w)\}$ , where

$$V_{\alpha}^{1}(x, w) = \overline{C_{\alpha}}((x, w), 1) + \sum_{(x,w)} P_{(x,w)(x,w)}(1) e^{-\alpha t} V((x, w)) dF_{(x,w)(x,w)}(t \mid 1)$$

$$= \frac{hx}{\alpha + \lambda(x, w)} + \frac{\lambda}{\alpha + \lambda(x, w)} V_{\alpha}(x + 1, w) + \frac{wf}{\alpha + \lambda(x, w)} V_{\alpha}(x, w - 1) + \frac{\mu \min(x, w)}{\alpha + \lambda(x, w)} V_{\alpha}(x - 1, w)$$

$$V_{\alpha}^{2}(x, w) = \frac{\frac{cr}{N - w}}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} [S + (N - w)r_{c}] + \frac{hx}{\alpha + \lambda(x, w) + \frac{cr}{N - w}}$$

$$+ \frac{\lambda}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} V_{\alpha}(x + 1, w) + \frac{wf}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} V_{\alpha}(x, w - 1) + \frac{\mu \min(x, w)}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} V_{\alpha}(x - 1, w)$$

$$+ \frac{\frac{cr}{N - w}}{\alpha + \lambda(x, w) + \frac{cr}{N - w}} V_{\alpha}(x, N) \}$$

$$(3.5)$$

The state space is uncountable, the one step  $cost \overline{C}((x,w),a)$  is not bounded. Using standard results from unbounded cost MDPs (Hernandez-Lerma(1996), Th. 4.2.3, Prop. 4.3.1, (c)), the following Lemma follows.

#### Lemma 3.1:

(i) V<sub>α</sub>(x, w) is the unique solution to the optimality equations in (3.5).
(ii) The policy that takes actions to minimize the right hand side of equation (3.5) is an optimal stationary policy.

(iii) Let  $V_{\alpha}(x, w; n = 0) = 0$ ;  $V_{\alpha}(x, w; n) = \min_{a \in A(x,w)} \{\overline{C}((x, w), a) + \overline{V}((x, w), a; n - 1)\};$ 

Then  $\lim_{x \to \infty} V_{\alpha}(x, w; n) = V_{\alpha}(x, w)$  for any initial state (x,w).

# 4. Modified Model by Uniformization

In Section 3 all transition rates are bounded by  $\alpha + \lambda + Nf + N\mu + cr$  in state (x,w), the process leaves this state with rate  $\lambda(x, w)$  when a=1, and  $\lambda(x, w) + \frac{cr}{N-w}$  when a=2. In this section, we employ a method, referred to in the literature as the uniformization approach (Puterman(1994) and Tijms(1994)) to transform the problem in Section 3 into an equivalent model where all sojourn times follow exponential distribution with the same rate  $\overline{\lambda} \quad \alpha + \lambda + Nf + N\mu + cr$ . This model is defined as follows:

Transitions out of each state occur at a constant rate  $\overline{\lambda}$ . When the process is in state (x,w), however, only a

fraction  $\frac{\lambda(x, w)}{\overline{\lambda}}$  or  $\frac{\lambda(x, w) + \frac{cr}{N-w}}{\overline{\lambda}}$  are those transitions from (x,w) to state (x,w)'  $\neq$  (x,w) and the rest are transitions back to state (x,w) (These are the "virtual transitions"). Specifically, define an equivalent discrete-time Markov chain of which transition mechanism are given by (1) When a=1 (continue)

(x,w) 
$$\rightarrow$$
 (x+1,w) with probability  $\frac{\lambda}{\overline{\lambda}}$ . (x,w)  $\rightarrow$  (x,w-1) with probability  $\frac{\text{wf}}{\overline{\lambda}}$ .  
(x,w)  $\rightarrow$  (x-1,w) with probability  $\frac{\mu \min(x, w)}{\overline{\lambda}}$ . (x,w)  $\rightarrow$  absorbing with probability  $\frac{\alpha}{\overline{\lambda}}$ .  
(x,w)  $\rightarrow$  (x,w) with probability  $(1 - \frac{\alpha + \lambda(x, w)}{\overline{\lambda}})$ . (virtual transitions)  
(2) When a=2 (do the group maintenance)  
(x,w)  $\rightarrow$  (x+1,w) with probability  $\frac{\lambda}{\overline{\lambda}}$ . (x,w)  $\rightarrow$  (x,w-1) with probability  $\frac{\text{wf}}{\overline{\lambda}}$ .  
(x,w)  $\rightarrow$  (x-1,w) with probability  $\frac{\mu \min(x, w)}{\overline{\lambda}}$ . (x,w)  $\rightarrow$  (x,N) with probability  $\frac{\nabla - w}{\overline{\lambda}}$ .  
(x,w)  $\rightarrow$  absorbing state with probability  $\frac{\alpha}{\overline{\lambda}}$ . (x,w)  $\rightarrow$  (x,w) with probability  $(1 - \frac{\alpha + \lambda(x, w) + \frac{\text{cr}}{N - w})}{\overline{\lambda}})$ .

By using this modified MDP, the optimality equation can be derived as follows:

Recall the optimality equation in (3.5), when action a=1, we have  $V_{\alpha}(x, w) = V_{\alpha}^{1}(x, w)$ . By multiplying both sides of the equation with  $\frac{\alpha + \lambda(x, w)}{\overline{\lambda}}$ , then adding both sides of the above equation with  $V_{\alpha}(x, w)$  and moving  $\frac{\alpha + \lambda(x, w)}{\overline{\lambda}} V_{\alpha}(x, w)$  from left side of equation to right side of equation, we get  $V_{\alpha}^{1}(x, w)$ 

$$=\frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} \nabla_{\alpha}(x+1,w) + \frac{wf}{\overline{\lambda}} \nabla_{\alpha}(x,w-1) + \frac{\mu\min(x,w)}{\overline{\lambda}} \nabla_{\alpha}(x-1,w) + (1 - \frac{\alpha + \lambda(x,w)}{\overline{\lambda}}) \nabla_{\alpha}(x,w) \quad (4.1)$$
When action a=2, we have  $V_{\alpha}(x,w) = V_{\alpha}^{2}(x,w)$ 

When action a=2, we have  $V_{\alpha}(x,w) = V_{\alpha}^2(x,w)$ .

After multiplying both sides of the above equation with  $\frac{\alpha + \lambda(x, w) + \frac{cr}{N - w}}{\overline{\lambda}}$ , then by adding both sides of the above

equation with  $V_{\alpha}(x, w)$  and moving  $\frac{\alpha + \lambda(x, w) + \frac{cr}{N-w}}{\overline{\lambda}} V_{\alpha}(x, w)$  from left side of equation to right side of equation,

we get

$$V_{\alpha}^{2}(x, w) = \frac{\frac{cr}{N-w}}{\overline{\lambda}} [S + (N-w)r_{c}] + \frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} V_{\alpha}(x+1, w) + \frac{wf}{\overline{\lambda}} V_{\alpha}(x, w-1) + \frac{\mu\min(x, w)}{\overline{\lambda}} V_{\alpha}(x-1, w) + \frac{\frac{cr}{N-w}}{\overline{\lambda}} V_{\alpha}(x, N) + (1 - \frac{\alpha + \lambda(x, w) + \frac{cr}{N-w}}{\overline{\lambda}}) V_{\alpha}(x, w)$$
(4.2)

From the results of (4.1) and (4.2), the modified optimality equations by Uniformization can be written as  $V_{\alpha}^{U}(x, w) = Min \{V_{\alpha}^{1}(x, w), V_{\alpha}^{2}(x, w)\}$  where

$$\begin{aligned}
\nabla_{\alpha}^{1}(x, w) &= \frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} \nabla_{\alpha}(x+1, w) + \frac{wf}{\overline{\lambda}} \nabla_{\alpha}(x, w-1) + \frac{\mu \min(x, w)}{\overline{\lambda}} \nabla_{\alpha}(x-1, w) + (1 - \frac{\alpha + \lambda(x, w)}{\overline{\lambda}}) \nabla_{\alpha}(x, w) \\
\nabla_{\alpha}^{2}(x, w) &= \frac{\frac{cr}{N-w}}{\overline{\lambda}} [S + (N-w)r_{c}] + \frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} \nabla_{\alpha}(x+1, w) + \frac{wf}{\overline{\lambda}} \nabla_{\alpha}(x, w-1) \\
&+ \frac{\mu \min(x, w)}{\overline{\lambda}} \nabla_{\alpha}(x-1, w) + \frac{\frac{cr}{N-w}}{\overline{\lambda}} \nabla_{\alpha}(x, N) + (1 - \frac{\alpha + \lambda(x, w) + \frac{cr}{N-w}}{\overline{\lambda}}) \nabla_{\alpha}(x, w) \end{aligned}$$
(4.3)

The optimality equations in (3.5) and (4.3) are equivalent.

In the sequel we will use optimality equation (4.3) in order to prove properties of the value function and the optimal policy. Now we define the successive approximation version of equation (4.3) as follows:  $V_{\alpha}$  (x, w;0) = 0,

$$\begin{aligned} \nabla_{\alpha}(\mathbf{x}, \mathbf{w}; \mathbf{n}) &= \operatorname{Min}\left\{\frac{h\mathbf{x}}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}+1, \mathbf{w}; \mathbf{n}-1) + \frac{\mathbf{w}f}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}, \mathbf{w}-1; \mathbf{n}-1) + \frac{\mu\min\left(\mathbf{x}, \mathbf{w}\right)}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}-1, \mathbf{w}; \mathbf{n}-1) \right. \\ &+ \left(1 - \frac{\alpha + \lambda(\mathbf{x}, \mathbf{w})}{\overline{\lambda}}\right)\nabla_{\alpha}(\mathbf{x}, \mathbf{w}; \mathbf{n}-1), \\ &\frac{\frac{cr}{\overline{N-w}}}{\overline{\lambda}}\left[S + (N-w)r_{c}\right] + \frac{h\mathbf{x}}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}+1, \mathbf{w}; \mathbf{n}-1) + \frac{wf}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}, \mathbf{w}-1; \mathbf{n}-1) + \frac{\mu\min\left(\mathbf{x}, \mathbf{w}\right)}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}-1, \mathbf{w}; \mathbf{n}-1) \\ &+ \frac{\frac{cr}{\overline{N-w}}}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n}-1) + \left(1 - \frac{\alpha + \lambda(\mathbf{x}, \mathbf{w}) + \frac{cr}{\overline{N-w}}}{\overline{\lambda}}\right)\nabla_{\alpha}(\mathbf{x}, \mathbf{w}; \mathbf{n}-1) \quad \text{for } \mathbf{n} \quad 1 \end{aligned}$$

# 5. The Properties of Optimal Policy

We first prove the monotonicity property for the optimal discounted cost function  $V_{\alpha}(x, w)$ , which will be used in proving the structure of optimal policy.

**Theorem 5.1**  $V_{\alpha}(x, w)$  is an increasing function of x.

#### **Proof:**

The proof will be by induction on n. For n=1, we have

$$V_{\alpha}(x, w; n = 1) = Min\{\frac{hx}{\overline{\lambda}}, \frac{cr}{\overline{N-w}}[S + (N - w)_{r_c}] + \frac{hx}{\overline{\lambda}}\} \text{ where } \overline{\lambda} = \alpha + \lambda + N\mu + Nf + cr$$

 $V_{\alpha}(x, w; n = 1)$  is increasing in x, because it is the minimum of two increasing functions of x. Assume that  $V_{\alpha}(x, w; n)$  is increasing in x, then we need to prove  $V_{\alpha}(x, w; n+1)$  is also increasing in x. (1)  $V_{\alpha}^{1}(x, w; n+1)$  is increasing in x.

From (4.4), we need to show that  $V^1(x, w; n+1) - V^1(x+1, w; n+1) = 0$ , for all w. Case 1: x w :

Then x+1>w and  $\min(x,w)=\min(x+1,w)=w$ , and the difference  $V^1(x, w; n+1) - V^1(x+1, w; n+1)$  is simplified as follows:

$$-\frac{h}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} [\nabla_{\alpha}(x+1,w;n) - \nabla_{\alpha}(x+2,w;n)] + \frac{wf}{\overline{\lambda}} [\nabla_{\alpha}(x,w-1;n) - \nabla_{\alpha}(x+1,w-1;n)] + \frac{wu}{\overline{\lambda}} [\nabla_{\alpha}(x-1,w;n) - \nabla_{\alpha}(x,w;n)] + (1 - \frac{\alpha+\lambda+wf+wu}{\overline{\lambda}}) [\nabla_{\alpha}(x,w;n) - \nabla_{\alpha}(x+1,w;n)]$$

Because all quantities in brackets are non-positive from the induction hypothesis, the above expression is non-positive. Case 2: x<w :

Then min(x,w)=x, min(x+1,w)=x+1, and  $V^1(x,w;n+1) - V^1(x+1,w;n+1)$ 

$$= -\frac{h}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} [\nabla_{\alpha}(x+1,w;n) - \nabla_{\alpha}(x+2,w;n)] + \frac{wf}{\overline{\lambda}} [\nabla_{\alpha}(x,w-1;n) - \nabla_{\alpha}(x+1,w-1;n)] + \frac{xu}{\overline{\lambda}} [\nabla_{\alpha}(x-1,w;n) - \nabla_{\alpha}(x,w;n)] + (1 - \frac{\alpha+\lambda+wf+(x+1)u}{\overline{\lambda}}) [\nabla_{\alpha}(x,w;n) - \nabla_{\alpha}(x+1,w;n)]$$

Since all terms in brackets are non-positive,  $\nabla^1(x, w; n+1) - \nabla^1(x+1, w; n+1) = 0$  for x<w. Now because  $\nabla^1(x, w; n+1) - \nabla^1(x+1, w; n+1) = 0$  for all x, we can conclude that  $\nabla^1(x, w; n+1)$  is increasing in x. (2)  $\nabla^2(x, w; n+1)$  is increasing in x.

(2)  $\nabla$  (x, w; n + 1) is increasing in x.

From (4.4), we need to show that  $\nabla^2(x, w; n+1) - \nabla^2(x+1, w; n+1) = 0$  for all w. Case 1: x w :

Then x+1>w and min(x,w)=min(x+1,w)=w, and

 $V^{2}(x, w; n+1) - V^{2}(x+1, w; n+1)$ 

$$= -\frac{h}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} [\nabla_{\alpha}(x+1,w;n) - \nabla_{\alpha}(x+2,w;n)] + \frac{wf}{\overline{\lambda}} [\nabla_{\alpha}(x,w-1;n) - \nabla_{\alpha}((x+1,w-1;n))] \\ + \frac{w\mu}{\overline{\lambda}} [\nabla_{\alpha}(x-1,w;n) - \nabla_{\alpha}(x,w;n)] + \frac{\frac{cr}{N-w}}{\overline{\lambda}} [\nabla_{\alpha}(x,N;n) - \nabla_{\alpha}(x+1,N;n)] \\ + (1 - \frac{\alpha+\lambda+wf+w\mu+\frac{cr}{N-w}}{\overline{\lambda}}) [\nabla_{\alpha}(x,w;n) - \nabla_{\alpha}(x+1,w;n)]$$

Because all quantities in brackets are non-positive from the induction hypothesis, the above expression is non-positive for  $x \ w$ .

Case 2: x<w:

Then min(x,w)=x, min(x+1,w)=x+1, and  $V^2(x, w; n+1) - V^2(x+1, w; n+1)$ 

$$= -\frac{h}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} [\nabla_{\alpha}(x+1,w;n) - \nabla_{\alpha}(x+2,w;n)] + \frac{wf}{\overline{\lambda}} [\nabla_{\alpha}(x,w-1;n) - \nabla_{\alpha}(x+1,w-1;n)]$$

$$+ \frac{xu}{\overline{\lambda}} [\nabla_{\alpha}(x-1,w;n) - \nabla_{\alpha}(x,w;n)] + \frac{\frac{cr}{N-w}}{\overline{\lambda}} [\nabla_{\alpha}(x,N;n) - \nabla_{\alpha}(x+1,N;n)]$$

$$+ (1 - \frac{\alpha + \lambda + wf + (x+1)u + \frac{cr}{N-w}}{\overline{\lambda}}) [\nabla_{\alpha}(x,w;n) - \nabla_{\alpha}(x+1,w;n)] = 0$$

Because all quantities in above brackets are non-positive from the induction hypothesis,  $V^2(x, w; n+1) - V^2(x+1, w; n+1)$  is non-positive for x<w.

Since  $V^2(x, w; n+1) - V^2(x+1, w; n+1) = 0$  for all x, we can conclude that  $V^2(x, w; n+1)$  is increasing in x.

(3)  $V_{\alpha}(x, w; n+1) = Min \{V^1(x, w; n+1), V^2(x, w; n+1)\}$  is the minimum of two increasing functions, and this concludes that  $V_{\alpha}(x, w; n+1)$  is also increasing in x, and this concludes the induction proof. Therefore,  $V_{\alpha}(x, w; n)$  is increasing in x for all n, and by using standard arguments from successive approximations for discounted Markovian Decision Processes,  $V_{\alpha}(x, w) = \lim_{\alpha \to \infty} V_{\alpha}(x, w; n)$  is also increasing in x. The proof is completed. 

In this section we derive the monotonicity properties for the difference function between  $V_{\alpha}^{1}(x,w)$  and  $V_{\alpha}^{2}(x,w)$ . These properties are important for providing the threshold structure of the optimal policy. Before this, we need to prove the following Theorem.

**Theorem 5.2**  $V_{\alpha}(x,w)$ -  $V_{\alpha}(x,N)$  is increasing in x.

### **Proof:**

Since  $V_{\alpha}(x,w) = \min\{V_{\alpha}^{1}(x,w), V_{\alpha}^{2}(x,w)\}\)$ , we consider two cases as follows:

(1) 
$$V_{\alpha}(x,w) = V_{\alpha}^{2}(x,w)$$
:

The proof will be by induction on n. From (4.4), for n=1, we have

$$V_{\alpha}(x, w; n = 1) - V_{\alpha}(x, N; n = 1) = V_{\alpha}^{2}(x, w; n = 1) - V_{\alpha}^{1}(x, N; n = 1) = \frac{\frac{cr}{N - w}}{\overline{\lambda}} [S + (N - w)_{r_{c}}]$$

It is obvious that  $V_{\alpha}(x, w; n = 1) - V_{\alpha}(x, N; n = 1)$  is non-decreasing in x.

We now assume that  $V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)$  is increasing in x, then we need to prove that  $V_{\alpha}(x,w;n+1) - V_{\alpha}(x,N;n+1)$  is also increasing in x. From (4.4), we have  $V_{\alpha}(x,w;n+1) - V_{\alpha}(x,N;n+1) = V_{\alpha}^{2}(x,w;n+1) - V_{\alpha}^{1}(x,N;n+1)$ 

$$=\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S+(N-w)r_{c}]+\frac{\lambda}{\overline{\lambda}}[V_{\alpha}(x+1,w;n)-V_{\alpha}(x+1,N;n)]+[\frac{\mu\min(x,w)}{\overline{\lambda}}V_{\alpha}(x-1,w;n)$$

$$-\frac{\mu\min(x, N)}{\overline{\lambda}}V_{\alpha}(x-1, N; n)] + \left[\frac{wf}{\overline{\lambda}}V_{\alpha}(x, w-1; n) - \frac{Nf}{\overline{\lambda}}V_{\alpha}(x, N-1; n)\right] + \frac{cr}{\frac{N-w}{\overline{\lambda}}}V_{\alpha}(x, N; n)$$

+ 
$$\left[\left(1 - \frac{\alpha + \lambda(x, w) + \frac{cr}{N - w}}{\overline{\lambda}}\right) V_{\alpha}(x, w; n) - \left(1 - \frac{\alpha + \lambda(x, N)}{\overline{\lambda}}\right) V_{\alpha}(x, N; n)\right]$$
 (5.1)

Now we consider three cases for x:

#### Case 1: x N

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In this case min(x,N)=N and min(x,w)=w, therefore (5.1) becomes

$$\begin{aligned} & \nabla_{\alpha}(\mathbf{x}, \mathbf{w}; \mathbf{n}+1) - \nabla_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n}+1) \\ &= \frac{cr}{\frac{N-w}{\overline{\lambda}}} [S + (N-w)_{r_c}] + \frac{\lambda}{\overline{\lambda}} [\nabla_{\alpha}(\mathbf{x}+1, \mathbf{w}; \mathbf{n}) - \nabla_{\alpha}(\mathbf{x}+1, \mathbf{N}; \mathbf{n})] + \frac{w\mu}{\overline{\lambda}} [\nabla_{\alpha}(\mathbf{x}-1, \mathbf{w}; \mathbf{n}) - \nabla_{\alpha}(\mathbf{x}-1, \mathbf{N}; \mathbf{n})] \end{aligned}$$

$$-\frac{(N-w)\mu}{\overline{\lambda}}V_{\alpha}(x-1,N;n)+\frac{wf}{\overline{\lambda}}[V_{\alpha}(x,w-1;n)-V_{\alpha}(x,N-1;n)]-\frac{(N-w)f}{\overline{\lambda}}V_{\alpha}(x,N-1;n)$$

$$+\frac{\frac{cr}{N-w}}{\overline{\lambda}}V_{\alpha}(x, N; n) + (1 - \frac{\alpha + \lambda + N\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)]$$

$$+\frac{(N-w)\mu}{\overline{\lambda}}V_{\alpha}(x, w; n) + \frac{(N-w)f}{\overline{\lambda}}V_{\alpha}(x, w; n) - \frac{\frac{cr}{N-w}}{\overline{\lambda}}V_{\alpha}(x, N; n)$$

$$=\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S+(N-w)_{rc}]+\frac{\lambda}{\overline{\lambda}}[V_{\alpha}(x+1,w;n)-V_{\alpha}(x+1,N;n)]+\frac{w\mu}{\overline{\lambda}}[V_{\alpha}(x-1,w;n)-V_{\alpha}(x-1,N;n)]$$

$$+\frac{(N-w)\mu}{\overline{\lambda}}[\nabla_{\alpha}(x, w; n) - \nabla_{\alpha}(x-1, N; n)] + \frac{wf}{\overline{\lambda}}[\nabla_{\alpha}(x, w-1; n) - \nabla_{\alpha}(x, N-1; n)]$$

$$+\frac{(N-w)f}{\overline{\lambda}}[\nabla_{\alpha}(x, w; n) - \nabla_{\alpha}(x, N-1; n)] + (1 - \frac{\alpha + \lambda + N\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[\nabla_{\alpha}(x, w; n) - \nabla_{\alpha}(x, N; n)]$$

After rearranging terms, we obtain  $V_{\alpha}(x, w; n+1) - V_{\alpha}(x, N; n+1)$ 

$$=\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S+(N-w)r_{c}]+\frac{\lambda}{\overline{\lambda}}[\nabla_{\alpha}(x+1,w;n)-\nabla_{\alpha}(x+1,N;n)]+\frac{w\mu}{\overline{\lambda}}[\nabla_{\alpha}(x-1,w;n)-\nabla_{\alpha}(x-1,N;n)]$$

$$+\frac{(N-w)\mu}{\overline{\lambda}}[\nabla_{\alpha}(x,w;n)-\nabla_{\alpha}(x,N;n)]+\frac{wf}{\overline{\lambda}}[\nabla_{\alpha}(x,w-1;n)-\nabla_{\alpha}(x,N;n)]$$

$$+\frac{(N-w)f}{\overline{\lambda}}[\nabla_{\alpha}(x,w;n)-\nabla_{\alpha}(x,N;n)]+\{(1-\frac{\alpha+\lambda+N\mu+Nf+\frac{cr}{N-w}}{\overline{\lambda}})[\nabla_{\alpha}(x,w;n)-\nabla_{\alpha}(x,N;n)]$$

$$-\frac{(N-w)\mu}{\overline{\lambda}}[\nabla_{\alpha}(x-1,N;n)-\nabla_{\alpha}(x,N;n)]-\frac{Nf}{\overline{\lambda}}[\nabla_{\alpha}(x,N-1;n)-\nabla_{\alpha}(x,N;n)]\}$$
(5.2)

According to the induction hypothesis,

The first six terms in (5.2) are clearly increasing in x. To finish this proof, we need to show  $\Delta^1(x, w; n+1)$  is increasing in x, where

$$\Delta^{1}(x, w; n+1) = (1 - \frac{\alpha + \lambda + N\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)] - \frac{(N-w)\mu}{\overline{\lambda}}[V_{\alpha}(x, -1, N; n) - V_{\alpha}(x, N; n)] - \frac{Nf}{\overline{\lambda}}[V_{\alpha}(x, N-1; n) - V_{\alpha}(x, N; n)]$$

Recall that parameter  $\overline{\lambda}$  used in the uniformization approach must be greater than  $\overline{\lambda} = \alpha + \lambda + Nf + N\mu + cr$ . Here we choose  $\overline{\lambda} \quad \alpha + \lambda + Mf + M\mu + cr$  where M is sufficiently large number. Then we rearrange  $\Delta^1(x, w; n+1)$  as follows:

$$\Delta^{1}(\mathbf{x},\mathbf{w};\mathbf{n}+1) = (1 - \frac{\alpha + \lambda + M\mu + Mf + \frac{cr}{N-w}}{\overline{\lambda}})[\nabla_{\alpha}(\mathbf{x},\mathbf{w};\mathbf{n}) - \nabla_{\alpha}(\mathbf{x},\mathbf{N};\mathbf{n})]$$

$$\frac{(M-N)\mu + (M-N)f}{\overline{\lambda}} [v_{\alpha}(\textbf{x}, \, \textbf{w}; n) - v_{\alpha}(\textbf{x}, \, N; n)]$$

$$-\frac{(N-w)\mu}{\overline{\lambda}}[\nabla_{\alpha}(x-1,N;n)-\nabla_{\alpha}(x,N;n)]-\frac{Nf}{\overline{\lambda}}[\nabla_{\alpha}(x,N-1;n)-\nabla_{\alpha}(x,N;n)]$$

In the above expression, the first term is increasing in x. The coefficient of the second increasing term  $\frac{(M-N)\mu + (M-N)f}{\overline{\lambda}}$ , because of large M, is arbitrarily close to 1, while the coefficient of the last three terms is arbitrarily small. Therefore, the entire expression should be increasing in x, for M sufficiently large. Therefore,  $\Delta^1(x,w;n+1)$  is increasing in x. Now we can conclude that  $V_{\alpha}(x,w;n+1) - V_{\alpha}(x,N;n+1)$  is increasing in x when x N.

# Case 2: w x < N

In this case min(x,N)=x and min(x,w)=w, therefore (5.1) becomes

 $V_{\alpha}(x, w; n+1) - V_{\alpha}(x, N; n+1)$ 

$$=\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S+(N-w)r_{c}]+\frac{\lambda}{\overline{\lambda}}[V_{\alpha}(x+1,w;n)-V_{\alpha}(x+1,N;n)]+\frac{w\mu}{\overline{\lambda}}[V_{\alpha}(x-1,w;n)-V_{\alpha}(x-1,N;n)]-\frac{(x-w)\mu}{\overline{\lambda}}V_{\alpha}(x-1,N;n)+\frac{wf}{\overline{\lambda}}[V_{\alpha}(x,w-1;n)-V_{\alpha}(x,N-1;n)]-\frac{(N-w)f}{\overline{\lambda}}V_{\alpha}(x,N-1;n)]$$

$$+\frac{\frac{cr}{N-w}}{\overline{\lambda}}V_{\alpha}(x, N; n) + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)]$$

$$+\frac{(x-w)\mu}{\overline{\lambda}}V_{\alpha}(x, w; n) + \frac{(N-w)f}{\overline{\lambda}}V_{\alpha}(x, w; n) - \frac{\frac{cr}{N-w}}{\overline{\lambda}}V_{\alpha}(x, N; n)$$

$$=\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S+(N-w)r_{c}]+\frac{\lambda}{\overline{\lambda}}[\nabla_{\alpha}(x+1,w;n)-\nabla_{\alpha}(x+1,N;n)]+\frac{w\mu}{\overline{\lambda}}[\nabla_{\alpha}(x-1,w;n)-\nabla_{\alpha}(x-1,N;n)]$$

$$+\frac{(\mathbf{x}-\mathbf{w})\boldsymbol{\mu}}{\overline{\lambda}}[\boldsymbol{\nabla}_{\boldsymbol{\alpha}}(\mathbf{x},\,\mathbf{w};\,\mathbf{n})-\boldsymbol{\nabla}_{\boldsymbol{\alpha}}(\mathbf{x}\,-1,\,\mathbf{N};\,\mathbf{n})]+\frac{\mathbf{w}f}{\overline{\lambda}}[\boldsymbol{\nabla}_{\boldsymbol{\alpha}}(\mathbf{x},\,\mathbf{w}\,-1;\,\mathbf{n})-\boldsymbol{\nabla}_{\boldsymbol{\alpha}}(\mathbf{x},\,\mathbf{N}\,-1;\,\mathbf{n})]$$

$$+\frac{(N-w)f}{\overline{\lambda}}[V_{\alpha}(x, w; n) - V_{\alpha}(x, N-1; n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)]$$

After rearranging terms, we obtain  $V_{\alpha}(x, w; n+1) - V_{\alpha}(x, N; n+1)$ 

$$= \frac{\frac{cr}{N-w}}{\overline{\lambda}} [S + (N-w)r_{c}] + \frac{\lambda}{\overline{\lambda}} [V_{\alpha}(x+1,w;n) - V_{\alpha}(x+1,N;n)] + \frac{w\mu}{\overline{\lambda}} [V_{\alpha}(x-1,w;n) - V_{\alpha}(x-1,N;n)] + \frac{(x-w)\mu}{\overline{\lambda}} [V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + \frac{wf}{\overline{\lambda}} [V_{\alpha}(x,w-1;n) - V_{\alpha}(x,N;n)] + \frac{(N-w)f}{\overline{\lambda}} [V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + \{(1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}})[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)]$$

$$-\frac{(\mathbf{x}-\mathbf{w})\boldsymbol{\mu}}{\overline{\lambda}}[\boldsymbol{V}_{\alpha}(\mathbf{x}-1,\mathbf{w};\mathbf{n})-\boldsymbol{V}_{\alpha}(\mathbf{x},\mathbf{N};\mathbf{n})]-\frac{\mathbf{N}\mathbf{f}}{\overline{\lambda}}[\boldsymbol{V}_{\alpha}\mathbf{N}-1;\mathbf{n})-\boldsymbol{V}_{\alpha}(\mathbf{x},\mathbf{N};\mathbf{n})].$$
(5.3)

According to the induction hypothesis, the first six terms in (5.3) are clearly increasing in x.

To finish this proof, we need to show  $\Delta^2(x,w;n+1)$  is increasing in x, where

$$\Delta^{2}(\mathbf{x}, \mathbf{w}; \mathbf{n}+1) = (1 - \frac{\alpha + \lambda + \mathbf{x}\mu + \mathbf{N}\mathbf{f} + \frac{\mathbf{c}\mathbf{r}}{\mathbf{N} - \mathbf{w}}}{\overline{\lambda}})[\mathbf{V}_{\alpha}(\mathbf{x}, \mathbf{w}; \mathbf{n}) - \mathbf{V}_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n})]$$
$$- \frac{(\mathbf{x} - \mathbf{w})\mu}{\overline{\lambda}}[\mathbf{V}_{\alpha}(\mathbf{x} - 1, \mathbf{N}; \mathbf{n}) - \mathbf{V}_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n})] - \frac{\mathbf{N}\mathbf{f}}{\overline{\lambda}}[\mathbf{V}_{\alpha}(\mathbf{x}, \mathbf{N} - 1; \mathbf{n}) - \mathbf{V}_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n})] - \frac{\frac{\mathbf{c}\mathbf{r}}{\mathbf{N} - \mathbf{w}}}{\overline{\lambda}}\mathbf{V}_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n})]$$

Here we use the same assumption  $\overline{\lambda} \quad \alpha + \lambda + Mf + M\mu + cr$  as case 1. Then we rearrange  $\Delta^2(x,w;n+1)$  as follows:

$$\Delta^{2}(\mathbf{x},\mathbf{w};\mathbf{n}+1) = (1 - \frac{\alpha + \lambda + M\mu + Mf + \frac{cr}{N-w}}{\overline{\lambda}})[\nabla_{\alpha}(\mathbf{x}, w;\mathbf{n}) - \nabla_{\alpha}(\mathbf{x}, N;\mathbf{n})]$$

$$\frac{(M-x)\mu + (M-N)f}{\overline{\lambda}} [V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)] - \frac{(x-w)\mu}{\overline{\lambda}} [V_{\alpha}(x-1, N; n) - V_{\alpha}(x, N; n)] - \frac{Nf}{\overline{\lambda}} [V_{\alpha}(x, N-1; n) - V_{\alpha}(x, N; n)] - \frac{Cr}{\overline{\lambda}} V_{\alpha}(x, N; n)$$

In the above expression, by the induction hypothesis, the first two terms are clearly increasing in x. Because of large M, the coefficient of the second term,  $\frac{(M-x)\mu + (M-N)f}{\overline{\lambda}}$  is arbitrarily close to 1, while the coefficients of the last three terms are arbitrarily small. Therefore,  $\Delta^2(x,w;n+1) - \Delta^2(x+1,w;n+1) = 0$ , for M sufficiently large.

Since  $\Delta^2(x,w;n+1) - \Delta^2(x+1,w;n+1) = 0$ , we can conclude that  $\Delta^2(x,w;n+1)$  is increasing in x.

 $\label{eq:constraint} \text{Therefore, } \ V_{\alpha}(x,\,w;n+1) \text{ - } V_{\alpha}(x,\,N;n+1) \ \text{ is increasing in } x \ \text{when } w \quad x < N \,.$ 

**Case 3:** x < w

$$\begin{aligned} & \nabla_{\alpha}(\mathbf{x}, \, \mathbf{w}; \mathbf{n} + 1) - \nabla_{\alpha}(\mathbf{x}, \, \mathbf{N}; \mathbf{n} + 1) \\ &= \frac{cr}{\overline{\lambda} - \mathbf{w}} [\mathbf{S} + (\mathbf{N} - \mathbf{w})_{r_{c}}] + \frac{\lambda}{\overline{\lambda}} [\nabla_{\alpha}(\mathbf{x} + 1, \, \mathbf{w}; \mathbf{n}) - \nabla_{\alpha}(\mathbf{x} + 1, \, \mathbf{N}; \mathbf{n})] + \frac{x\mu}{\overline{\lambda}} [\nabla_{\alpha}(\mathbf{x} - 1, \, \mathbf{w}; \mathbf{n}) - \nabla_{\alpha}(\mathbf{x} - 1, \, \mathbf{N}; \mathbf{n})] \end{aligned}$$

$$+\frac{\mathrm{wf}}{\overline{\lambda}}[\mathrm{V}_{\alpha}(\mathbf{x}, \mathrm{w} - 1; \mathrm{n}) - \mathrm{V}_{\alpha}(\mathbf{x}, \mathrm{N} - 1; \mathrm{n})] - \frac{(\mathrm{N} - \mathrm{w})f}{\overline{\lambda}} \mathrm{V}_{\alpha}(\mathbf{x}, \mathrm{N} - 1; \mathrm{n})$$

$$+\frac{\frac{cr}{N-w}}{\overline{\lambda}}V_{\alpha}(x, N; n) + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)]$$

+ 
$$\frac{(N-w)f}{\overline{\lambda}}V_{\alpha}(x, w; n) - \frac{\frac{cr}{N-w}}{\overline{\lambda}}V_{\alpha}(x, N; n)$$

$$=\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S+(N-w)_{rc}]+\frac{\lambda}{\overline{\lambda}}[V_{\alpha}(x+1,w;n)-V_{\alpha}(x+1,N;n)]$$

$$+\frac{x\mu}{\overline{\lambda}}[\nabla_{\alpha}(x-1,w;n)-\nabla_{\alpha}(x-1,N;n)]+\frac{wf}{\overline{\lambda}}[\nabla_{\alpha}(x,w-1;n)-\nabla_{\alpha}(x,N-1;n)]$$

$$+\frac{(N-w)f}{\overline{\lambda}}[\nabla_{\alpha}(x, w; n) - \nabla_{\alpha}(x, N-1; n)] + (1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[\nabla_{\alpha}(x, w; n) - \nabla_{\alpha}(x, N; n)]$$

After rearranging terms, we obtain  $V_{\alpha}(x, w; n+1) - V_{\alpha}(x, N; n+1)$ 

$$= \frac{\frac{cr}{N-w}}{\overline{\lambda}} [S + (N - w)_{r_c}] + \frac{\lambda}{\overline{\lambda}} [V_{\alpha}(x + 1, w; n) - V_{\alpha}(x + 1, N; n)]$$
$$+ \frac{x\mu}{\overline{\lambda}} [V_{\alpha}(x - 1, w; n) - V_{\alpha}(x - 1, N; n)] + \frac{wf}{\overline{\lambda}} [V_{\alpha}(x, w - 1; n) - V_{\alpha}(x, N; n)]$$

$$+\frac{(N-w)f}{\overline{\lambda}}[V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)] + \{(1 - \frac{\alpha + \lambda + x\mu + Nf + \frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)] - \frac{Nf}{\overline{\lambda}}[V_{\alpha}(x, N-1; n) - V_{\alpha}(x, N; n)]$$
(5.4)

According to the induction hypothesis, The first five terms in (5.4) are increasing in x. To finish the proof, we need to show  $\Delta^3(x,w;n+1)$  is increasing in x, where  $\Delta^3(x,w;n+1)$ 

$$=(1-\frac{\alpha+\lambda+x\mu+Nf+\frac{cr}{N-w}}{\overline{\lambda}})[V_{\alpha}(x, w; n)-V_{\alpha}(x, N; n)]-\frac{Nf}{\overline{\lambda}}[V_{\alpha}(x, N-1; n)-V_{\alpha}(x, N; n)]$$

We first use the same assumption  $\overline{\lambda} = \alpha + \lambda + Mf + M\mu + cr$  as case 1, then we rearrange

$$\Delta^{3}(x,w;n+1) = (1 - \frac{\alpha + \lambda + x\mu + Mf + \frac{cr}{N-w}}{\overline{\lambda}})[\nabla_{\alpha}(x,w;n) - \nabla_{\alpha}(x,N;n)]$$

$$+\frac{(M-N)f}{\overline{\lambda}}[V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)] - \frac{Nf}{\overline{\lambda}}[V_{\alpha}(x, N-1; n) - V_{\alpha}(x, N; n)]$$

In the above expression, by the induction hypothesis, the first two terms are clearly increasing in x. Because of large M, the coefficient of the second term,  $\frac{(M - N)f}{\overline{\lambda}}$  is arbitrarily close to 1, while the coefficients of the last term is arbitrarily small.

Therefore,  $\Delta^3(x,w;n+1) - \Delta^3(x+1,w;n+1) = 0$ , for M sufficiently large. Since  $\Delta^3(x,w;n+1) - \Delta^3(x+1,w;n+1) = 0$ , we can conclude  $\Delta^3(x,w;n+1)$  is increasing in x. Therefore,  $\nabla_{\alpha}(x,w;n+1) - \nabla_{\alpha}(x,N;n+1)$  is increasing in x when x < w.

Depending on the results in above three cases, we can conclude that

 $V_{\alpha}(x, w; n+1) - V_{\alpha}(x, N; n+1)$  is increasing in x when  $V_{\alpha}(x, w; n+1) = V_{\alpha}^2(x, w; n+1)$ 

(2) 
$$V_{\alpha}(x,w) = V_{\alpha}^{1}(x,w)$$
:

The proof will be by induction on n. From (4.4), for n=1, we have

$$V_{\alpha}(x, w; n = 1) - V_{\alpha}(x, N; n = 1) = \frac{hx}{\overline{\lambda}} - \frac{hx}{\overline{\lambda}} = 0$$

It is obvious that  $V_{\alpha}(x, w; n = 1) - V_{\alpha}(x, N; n = 1)$  is non-decreasing in x.

From the above result, we can assume that  $V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)$  is increasing in x, then we need to prove  $V_{\alpha}(x,w;n+1) - V_{\alpha}(x,N;n+1)$  is also increasing in x. From (4.4), we have

$$\begin{aligned} & V_{\alpha}(\mathbf{x}, \mathbf{w}; \mathbf{n}+1) - V_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n}+1) = \left[\frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} V_{\alpha}(\mathbf{x}+1, \mathbf{w}; \mathbf{n}) + \frac{wf}{\overline{\lambda}} V_{\alpha}(\mathbf{x}, \mathbf{w}-1; \mathbf{n}) \right. \\ & + \frac{\mu \min\left(\mathbf{x}, \mathbf{w}\right)}{\overline{\lambda}} V_{\alpha}(\mathbf{x}-1, \mathbf{w}; \mathbf{n}) + \left(1 - \frac{\alpha + \lambda(\mathbf{x}, \mathbf{w})}{\overline{\lambda}}\right) V_{\alpha}(\mathbf{x}, \mathbf{w}; \mathbf{n})\right] - \left[\frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} V_{\alpha}(\mathbf{x}+1, \mathbf{N}; \mathbf{n}) \right. \\ & + \frac{\mu \min\left(\mathbf{x}, \mathbf{N}\right)}{\overline{\lambda}} V_{\alpha}(\mathbf{x}-1, \mathbf{N}; \mathbf{n}) + \frac{Nf}{\overline{\lambda}} V_{\alpha}(\mathbf{x}, \mathbf{N}-1; \mathbf{n}) + \left(1 - \frac{\alpha + \lambda(\mathbf{x}, \mathbf{N})}{\overline{\lambda}}\right) V_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n})\right] \\ & = \frac{\lambda}{\overline{\lambda}} \left[ V_{\alpha}(\mathbf{x}+1, \mathbf{w}; \mathbf{n}) - V_{\alpha}(\mathbf{x}+1, \mathbf{N}; \mathbf{n}) \right] + \left[ \frac{\mu \min\left(\mathbf{x}, \mathbf{w}\right)}{\overline{\lambda}} V_{\alpha}(\mathbf{x}-1, \mathbf{w}; \mathbf{n}) - \frac{\mu \min\left(\mathbf{x}, \mathbf{N}\right)}{\overline{\lambda}} V_{\alpha}(\mathbf{x}, \mathbf{N}-1; \mathbf{n}) \right] \\ & + \left[ \frac{wf}{\overline{\lambda}} V_{\alpha}(\mathbf{x}, \mathbf{w}-1; \mathbf{n}) - \frac{Nf}{\overline{\lambda}} V_{\alpha}(\mathbf{x}, \mathbf{N}-1; \mathbf{n}) \right] + \left[ \left(1 - \frac{\alpha + \lambda(\mathbf{x}, \mathbf{w})}{\overline{\lambda}}\right) V_{\alpha}(\mathbf{x}, \mathbf{w}; \mathbf{n}) - \left(1 - \frac{\alpha + \lambda(\mathbf{x}, \mathbf{N})}{\overline{\lambda}}\right) V_{\alpha}(\mathbf{x}, \mathbf{N}; \mathbf{n}) \right] \end{aligned}$$

The above equation (5.5) is similar to the equation (5.1) from case (1), but with three terms  $(\frac{cr}{\overline{\lambda}-w}[S-(N-w)r_c], \frac{cr}{\overline{\lambda}}V_{\alpha}(x,N;n) \text{ and } -\frac{cr}{\overline{\lambda}}V_{\alpha}(x,w;n))$  short. Although these differences, we can still apply the similar deriving procedures in case (1) with little modification to prove equation (5.5) is increasing in x. Therefore,  $V_{\alpha}(x, w; n+1) - V_{\alpha}(x, N; n+1)$  is increasing in x when  $V_{\alpha}(x, w; n+1) = V_{\alpha}^{1}(x, w; n+1)$ .

n)]

According the results of case (1) and case (2), we can conclude that  $V_{\alpha}(x, w; n+1) - V_{\alpha}(x, N; n+1)$  is increasing in x. By using standard arguments from successive approximation for discounted MDPs,  $V_{\alpha}(x, w) - V_{\alpha}(x, N) = \lim_{n} \{V_{\alpha}(x, w; n) - V_{\alpha}(x, N; n)\}$  is certainly increasing in x. The proof is complete.  $\Box$ 

**Theorem 5.3**  $V^1_{\alpha}(x,w)$ - $V^2_{\alpha}(x,w)$  is increasing in x.

### **Proof:**

The proof will be by induction on n.

From (4.4), for n=1, we have  $V_{\alpha}^{1}(x, w; n = 1) - V_{\alpha}^{2}(x, w; n = 1) = -\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S + (N-w)r_{c}]$ 

It is obvious that  $V^1_{\alpha}(x,w;n=1) - V^2_{\alpha}(x,w;n=1)$  is non-decreasing in x.

From the above result, we can assume that  $V_{\alpha}^{1}(x,w;n) - V_{\alpha}^{2}(x,w;n)$  is increasing in x, then we need to prove  $V_{\alpha}^{1}(x,w;n+1) - V_{\alpha}^{2}(x,w;n+1)$  is also increasing in x. From (4.4), we have

$$V_{\alpha}^{1}(x,w;n+1) - V_{\alpha}^{2}(x,w;n+1) = -\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S + (N-w)_{r_{c}}] + \frac{\frac{cr}{N-w}}{\overline{\lambda}}[V_{\alpha}(x,w;n) - V_{\alpha}(x,N;n)]$$

According the above result and Theorem 5.2, we can conclude that  $V_{\alpha}^{1}(x,w;n+1) - V_{\alpha}^{2}(x,w;n+1)$  is increasing in x.

By using standard arguments from successive approximations for discounted MDPs,  $V^{1}_{\alpha}(x, w) - V^{2}_{\alpha}(x, w) = \lim_{n} \{V^{1}_{\alpha}(x, w; n) - V^{2}_{\alpha}(x, w; n)\}$  is increasing in x. The proof is complete.  $\Box$ 

**Theorem 5.4** There are thresholds  $x^*(w) = 0$  such that when the system is in state (x,w) an á-optimal policy do the group replaces if and only if  $x = x^*(w)$ .

## **Proof:**

From (4.4), we have

$$\nabla_{\alpha}(\mathbf{x}, \mathbf{w}) = \operatorname{Min}\{\frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}+1, \mathbf{w}) + \frac{wf}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}, \mathbf{w}-1) + \frac{\mu\min(\mathbf{x}, \mathbf{w})}{\overline{\lambda}}\nabla_{\alpha}(\mathbf{x}-1, \mathbf{w}) + (1 - \frac{\alpha + \lambda(\mathbf{x}, \mathbf{w})}{\overline{\lambda}})\nabla_{\alpha}(\mathbf{x}, \mathbf{w}),$$

$$\frac{\frac{cr}{N-w}}{\overline{\lambda}}[S+(N-w)r_{c}]+\frac{hx}{\overline{\lambda}}+\frac{\lambda}{\overline{\lambda}}V_{\alpha}(x+1,w)+\frac{wf}{\overline{\lambda}}V_{\alpha}(x,w-1)+\frac{\mu\min(x,w)}{\overline{\lambda}}V_{\alpha}(x-1,w)+\frac{\frac{cr}{N-w}}{\overline{\lambda}}V_{\alpha}(x,N)$$

+ 
$$(1 - \frac{\alpha + \lambda(x, w) + \frac{cr}{N - w}}{\overline{\lambda}}) V_{\alpha}(x, w)$$
. Let

$$x^{*}(w) = Min\{x: \frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} \nabla_{\alpha}(x+1,w) + \frac{wf}{\overline{\lambda}} \nabla_{\alpha}(x,w-1) + \frac{\mu\min(x,w)}{\overline{\lambda}} \nabla_{\alpha}(x-1,w) + (1 - \frac{\alpha + \lambda(x,w)}{\overline{\lambda}}) \nabla_{\alpha}(x,w) \\ > \frac{cr}{\overline{\lambda} - w} [S + (N - w)r_{c}] + \frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} \nabla_{\alpha}(x+1,w) + \frac{wf}{\overline{\lambda}} \nabla_{\alpha}(x,w-1) + \frac{\mu\min(x,w)}{\overline{\lambda}} \nabla_{\alpha}(x-1,w) + \frac{\frac{cr}{N-w}}{\overline{\lambda}} \nabla_{\alpha}(x,N)$$

+ 
$$(1 - \frac{\alpha + \lambda(x, w) + \frac{cr}{N - w}}{\overline{\lambda}}) V_{\alpha}(x, w)$$

Now, by Theorem 5.3, it follows that  $V^1_{\alpha}(x,w) - V^2_{\alpha}(x,w)$  is increasing in x, and hence we have

$$V_{\alpha}(x, w) = \begin{array}{cc} V_{\alpha}^{1}(x, w) & \text{for } x < x^{*}(w) \\ V_{\alpha}^{2}(x, w) & \text{for } x & x^{*}(w) \end{array} \text{ where}$$

$$V_{\alpha}^{1}(x, w) = \frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} V_{\alpha}(x+1, w) + \frac{wf}{\overline{\lambda}} V_{\alpha}(x, w-1) + \frac{\mu \min(x, w)}{\overline{\lambda}} V_{\alpha}(x-1, w) + (1 - \frac{\alpha + \lambda(x, w)}{\overline{\lambda}}) V_{\alpha}(x, w)$$

$$V_{\alpha}^{2}(x, w) = \frac{\overline{N-w}}{\overline{\lambda}} [S + (N-w)r_{c}] + \frac{hx}{\overline{\lambda}} + \frac{\lambda}{\overline{\lambda}} V_{\alpha}(x+1, w) + \frac{wf}{\overline{\lambda}} V_{\alpha}(x, w-1) + \frac{\mu \min(x, w)}{\overline{\lambda}} V_{\alpha}(x-1, w)$$

$$+\frac{\frac{cr}{N-w}}{\overline{\lambda}}V_{\alpha}(x, N) + (1 - \frac{\alpha + \lambda(x, w) + \frac{cr}{N-w}}{\overline{\lambda}})V_{\alpha}(x, w)$$
. The proof is completed.  $\Box$ 

## 6. Conclusions and Extensions

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In this paper we have developed the problem formulation as a continuous time Markovian decision process, and derived the optimality equations for the discounted cost model. For simplifying the analysis process of this model, we have transformed **it** into an equivalent model where all sojourn times follow exponential distribution with the same rate by employing the uniformization approach. In the analysis of threshold policy structure, we have proved the following main properties.

 $V_{\alpha}(x, w)$  is an increasing function in x.

- $V^1_{\alpha}(x, w) V^2_{\alpha}(x, w)$  is increasing in x.
- There are thresholds  $x^*(w) = 0$  such that when the system is in state (x,w) a group maintenance is performed if and only if  $x = x^*(w)$ .

In this model we assume the positive maintenance time and clearly show that the optimal group maintenance policy has a threshold structure. It is also possible to consider extensions which perform positive maintenance but not allow server failures during maintenance.

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