Optimal Production Planning of Multi-Period Multi-Product System: Mathematical Model and Spreadsheet Solution

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ABSTRACT This paper considers the production-inventory planning of multi-production manufacturing system over multi-period planning horizon. Some improvements are given on the basis of the conventional MPMP models and spreadsheet solution is also provided to support the implementation of such models in the management. This paper then analyzes the sensitivity of the optimal solution based on spreadsheet in detail.

INTRODUCTION

The “Multi-Period Multi-Product (MPMP) Production Planning System” problem is well known in the literature. The problem essentially consists of matching production levels of individual products to fluctuations of demand for a number of periods into the future, subject to constraints of capacities [1]. In this paper, the problem of MPMP production planning is further discussed, and we provide some improvements and solution suggestions available to the production planners. We assume in this paper that machines are reliable and completely flexible to produce any product at any fraction of its capacity. There is no cost when a machine switches from producing one product to producing another.

Our motivation of studying such production-planning problem lies on following aspects. First, the objective function about production planning in most of the academic papers is to minimize the total costs occurred during the production processes. We point out, however, that besides the cost information, the managers and planners in the manufacturing corporations may care more about the profit results based on the sales revenue, production costs, and various production constraints. Therefore our objective function of the model in this paper is to maximize the total profit.

Second, it has been a troublesome problem for the planners to optimize the MPMP production planning problems. Fortunately, since the spreadsheet software, such as Microsoft® Excel, is widely used in many corporations, we can use the Solver built in such software to solve production planning problems. And more powerful add-in Solver software can be easily ordered from the Frontline Systems Corporation. The availability and convenience of such solution method provide decision makers with obvious optimal solutions.

Generally speaking, it is inevitable that reality will deviate from the plan so that the deviation has to be controlled and the plan has to be modified if the discrepancy is detected. A rolling planning method is generally applied for supervising the newest market information, satisfying customer requirements, and maintaining the lowest inventory [3]. For multi-period problems, such dynamic planning method is especially important. We will discuss rolling planning finally and further apply it into the spreadsheet solution for dynamic planning.

CONSTRUCTION OF MPMP PLANNING MODEL

The MPMP production system considered in this paper is a make-to-inventory factory. That is, the company produces mainly according to the demand forecast. All finished products are put into the warehouse and customers pick up the goods from the warehouse. In this paper, the company wants to determine the number of workers needed each period, the number of overtime hours to
be used, the number of units to be produced, and the total profit associated with the plan.

In order to formulate the problem mathematically, the following notations are introduced.

Indexes

\( t \)  
The planning horizon time period, \( t=1,2,\ldots,T \)

\( p \)  
The type of products, \( p=1,2,\ldots,P \)

Parameters

\( C_h, C_f \)  
Hiring and Firing costs per worker

\( W_r, W_o \)  
Regular and Overtime wages per hour

\( MC, SC \)  
Machine and Storage Capacity-unit

\( HWM, OHWM \)  
Working and overtime hours per worker per month

\( EP_p, BC_p \)  
Expected price and backorder cost for product \( p \) per unit

\( TEI_p, TEW \)  
Target ending inventory for product \( p \) and Target ending workers

\( BI_p, BNW \)  
Beginning inventory for product \( p \) and Beginning number of workers

\( IC_p \)  
Inventory carrying cost for product \( p \) per unit per month

\( FD_{tp} \)  
Forecast demand for product \( p \) at period \( t \)

\( LHU_p \)  
Labor hours for product \( p \) per unit

Process variables

\( WA_t \)  
Workers available at period \( t \)

\( NL_{tp} \)  
Net Leftover for product \( p \) at period \( t \)

Decision variables

\( WH_t, WL_t \)  
Workers hired and laid off at period \( t \)

\( UP_{tp}, INV_{tp}, BO_{tp} \)  
Units produced, inventory and backorder for product \( p \) at period \( t \)

\( OHU_t \)  
Overtime hours used at period \( t \)

The MPMP production planning system can be expressed as following linear programming:

Max. \( Z = \sum_{t=1}^{T} \sum_{p=1}^{P} EP_p \sum_{i=1}^{T} \left( INV_{(t-1)p} + UP_{tp} - INV_{tp} \right) - C_h \sum_{t=1}^{T} WH_t - C_f \sum_{t=1}^{T} WL_t - W_o \sum_{t=1}^{T} OHU_t \)

\[ -W_r \sum_{t=1}^{T} HWM \left( WA_{t-1} + WH_t - WL_t \right) - \sum_{t=1}^{T} \sum_{p=1}^{P} IC_p \sum_{i=1}^{P} INV_{tp} - \sum_{t=1}^{T} \sum_{p=1}^{P} BC_p \sum_{i=1}^{P} BO_{tp} \]

st. \( OHU_t \leq OHWM \times WA_t, (2) \quad WA_0 = BNW, (3) \quad WA_t = WA_{t-1} + WH_t - WL_t, (4) \)

\[ \sum_{p=1}^{P} UP_{tp} \leq MC, (5) \quad \sum_{p=1}^{P} UP_{tp} \times LHU_p \leq OHU_t + HWM \times WA_t, (6) \quad \sum_{p=1}^{P} NL_{tp} \leq SC, (7) \]

\[ NL_{tp} = INV_{tp} - BO_{tp}, (8) \quad NL_{tp} = BI_p, (9) \quad NL_{tp} = UP_{tp} - FD_{tp} + NL_{(t-1)p}, (10) \]

\[ UP_{tp} - FD_{tp} + NL_{(t-1)p} = TEI_p, (11) \quad WA_p = TEW, (12) \]

\( WA_t, WL_t, OHU_t, UP_{tp}, INV_{tp}, BO_{tp} \geq 0, \quad WA_t \) and \( WL_t \) are integers.

The objective function is to maximize the sum of production profits at every period on the planning horizon. The relative constraints of production activities are explained as follows. Eq. (2) guarantees overtime hours used at period \( t \) less than or equal to the maximum overtime hours available at period \( t \). Eq. (3) initiates the number of available workers and Eq. (4) is the equilibrium equation of available workers that balances the workers available at period \( t-1 \), workers hired and laid off at period \( t \). Eq. (5) and (6) express the production constraints of machine capacity and labor capacity. Eq. (7) expresses the storage capacity for product \( p \) at period \( t \). Eq. (8) represents that at period \( t \) the number of net leftover for product \( p \) must be equal to the ending inventory of \( p \). Eq. (9) initiates the numbers of net leftovers and Eq. (10) is the inventory equilibrium equation that balances the inventory levels (including backorders) of product \( p \) at
period $t-1$. Quantities produced and demanded of product $p$ at period $t$. Eq. (11) and (12) represent that the ending inventory for product $p$ at the last period must be equal to the target inventory and the workers available at the last period must be equal to the target number of workers.

**SPREADSHEET SOLUTION WITH CASE STUDY**

Based on the mathematical model described above, we can find an optimal solution for such MPMP production-planning problem by means of spreadsheet modeling technique. Spreadsheet modeling is the process of entering the inputs and decision variables into a spreadsheet and then relating them appropriately, by means of formulas, to obtain the outputs [4]. In this paper, the inputs are the decision variables and various capacity constraints, and the outputs are the production plan and other information.

For example, the ABC Company is to development the production plan of its three products for the next six months. The demand data from forecast and input data are shown in Figure 1 and Figure 2. Under the conditions stated above, Figure 3 shows summary of the optimal solution for this problem by the Solver.

The optimal production quantities and inventory levels for three products are shown in the three charts of Figure 4. For product 1, the recommended production quantity in each month is
exactly equal to the demands of other two products are all fairly low at the same time, the production process of product 3 can gain enough resources to meet the demand requirements. Thus the inventory profile of product 3 is never below zero. Under such production quantities, the number of workers remain constant, 18 workers throughout the whole planning horizon, and there is no hiring or layoff costs. Total cost for this plan is $177,667 and total revenue is $236,250. Thus the planned total profit is $58,583.

**SENSITIVITY ANALYSIS**

Sensitivity analysis is the third stage of solving a problem completely. In Excel, there are several ways to perform a sensitivity analysis or to get a sensitivity report. If the Solver finds the optimal solution for a problem, it displays message in Solver Results dialog and we can obtain a new sheet with a lot of information about the model’s sensitivity to various inputs by checking the Sensitivity Report option. However sometimes the Solver’s sensitivity report is virtually impossible to unravel and it is believed to be more likely to confuse than to enlighten. Fortunately, Wayne L. Winston and S. Christian Albright [4] have written an add-in to Excel called SolverTable that makes sensitivity analysis much more straightforward. The SolverTable is to rerun the Solver for every new input or pair of inputs and then get the optimal solution for every scenario. There are two ways it can use, one-way table and two-way table. “One-way” table means that there is a single input and any number of output cells, and “two-way” table means that there are two input cells and one or more inputs. With SolverTable, this paper selects machine and storage capacity and backorder costs as examples to perform sensitivity analysis.

Machine capacity and storage capacity are regarded as two of the main constraints. Machine capacity causes a production plan to become more level, while storage capacity makes a variable production plan more desirable [2]. Figure 5 shows the effect of machine capacity constraint. In the data table, we set the minimum of machine capacity as 50, maximum as 200, and increment as 10, and the last row of data is the current optimal solution of this production system. Then we find that the production quantities and total profit of each product are identical once the machine capacity reaches 130 and conversely total costs is even more than total revenue if the machine capacity is below 90. We can learn these characteristics more clearly from the chart in Figure 5 and we find the break-even point of machine capacity is about 95 if other parameters remain fixed. In the optimal solution as shown in Figure 3, the inventory levels of all products are around zero. Therefore, the storage requirements of this problem is fairly low, and even if we set the storage capacity as zero, the total profit can reach 57,713, 98.5% of the current optimal profit.

We can also do a sensitivity analysis for backorder costs. Because the assumption of this problem is that the backorder costs for different product can be different, this sensitivity analysis will be more complex. Firstly, We analyze the sensitivity of total profit to unit backorder cost of three products independently. As shown in Figure 6, the common characteristic among the three products is that the total profit will increase if one’s unit backorder cost decreases to some degree. That is understandable since if one product’s unit backorder cost decreases to enough low, there will exist more resources to produce other products whose backorder costs are higher. Moreover, if unit backorder cost of product 2 decreases to $80 or even low level, the total profit can reach $177,667, which is within the range of sensitivity analysis.

![Figure 5: Machine Capacity Constraint](image_url)
higher than the profit reached by decreasing of one of other two products to same level. That is because the labor hours used per unit of product 2 is 30, 50% more than that of other two products, which means the labor hours to produce two units of product 2 can be used to produce three units of product 1 or 3 if there is no other constraint. And it is also because their labor hours used per unit are same that the sensitivity profiles of product 1 and product 3 are almost identical. Secondly, we analyze the two-way sensitivity of total profit to two unit backorder costs by means of two-way SolverTable. To explain this sensitivity more clear, we take the unit backorder costs of product 1 and product 2 as two dimensions. We can learn that the profit profile shows the laddered characteristic, and when unit backorder cost of product 2 is $10 the Solver gets the biggest optimal profit, $60,003. Through the ladders in both dimensions, the optimal profit will remain fixed in $58,500 when the unit backorder costs of product 1 and 2 respectively reach $70 and $90 or more. The similar sensitivity analysis of total profit to unit backorder costs of product 1 and 3 or product 2 and 3 can be easily done in the same way.

**FURTHER DISCUSSION: ROLLING PLANNING IN SPREADSHEET**

In reality, the planning models are usually implemented on a rolling planning horizon. The method of rolling planning is an implementation of plan-control-revision interaction process. The planning horizon is divided into several periods. At the beginning of first period, an initial plan is made to cover the whole horizon. But only the first period, also called frozen period, is put into practice. In next period, a new plan based on the previous one is produced considering the new developments in the first period. The planning horizon overlaps with previous one but reaches one period further.

This idea can also be put into spreadsheet modeling. In our example, if we observe month 1’s actual demand, what we need to do is to update the beginning inventory and number of workers in the Input Data area and the forecast demand for month 2 to month 7. The beginning worker number of new planning horizon is equal to the number of workers available in month 1, and the new beginning inventory is equal to the sum of net leftover before month 1 and unit produced in month 1 minus the actual demand in month 1. If necessary, we can also update the other parameters at the same time. We can rerun the Solver and the planning horizon goes rolling along!

**REFERENCES**


