ANALYTICAL PRICING AMERICAN CALL OPTION WITH KNOWN DIVIDEND

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ABSTRACT

The valuation of American options on dividend-paying asset is an important problem in financial economics. American options provide early exercise opportunities to pose the additional difficulty to obtain the closed-form solution. In this study, a recursive formula is developed for determining the optimal exercise price of American call options with known dividend based on the backward dynamic programming recursions and the Black-Scholes model combined with the Martingale Pricing and the Girsanov Theorem under the risk-neutral measure. Closed-form solution for American call option is obtained by taking the risk-neural expectation of its payoff and discounting it to the current. Previous attempts at pricing these options have been accurate but computationally expensive. This paper provides a simple, and an inexpensive solution for pricing American call option.

KEYWORD: American option pricing, Martingale Pricing, Girsanov Theorem, backward dynamic programming recursions

1. INTRODUCTION

An option is a security which gives its owner the right to trade in a fixed number of shares of a specified common stock at a fixed price at any time on or before a given date (Black and Scholes, 1973; Cox et al., 1979). A European option can be exercised only at the end of its life; an American option

can be exercised at any time during its life (Hull, 2003). This option written on a wide variety of commodities and commodity futures contract now trade in the U.S. and Canada. An American option has early exercise premiums implicitly embedded in their prices (Barone-Adesi and Whaley, 1987). Since the holder of American option can choose the exercise time, it is more complicated to analyze this situation than its European counterpart (Yang, 2002). The solution to the American option is more challenging and to date there is non-existing closed-form model available for practitioners in the field of finance (Hon, 2002; Yang, 2002; Underwood and Wang, 2002). It is well known that the American option pricing can be treated as a free boundary problem in which no analytical formula is available. Until recently, there were a number of different numerical methods for valuation of the American option. For instance, the finite difference method (Brennan and Schwartz, 1978) was introduced to price American option, and Cox et al. (1979) proposed the binomial method for valuation American option. These methods discretize both time and the state spaces in order to approximate the option price. The front-fixing finite difference method (Wu and Kwok, 1997), the Monte Carlo simulation (Grant et al., 1996), and the integral equation method (Huang et al., 1996) are used to approximate the American option. A comparison of these numerical methods can be found in review papers (Geske and Shastri, 1985; Broadie and Detemple, 1996). Some kinds of quasi-analytical formulas for valuation of the American option have been proposed (Johnson, 1983; Geske and Johnson, 1984; MacMillan, 1986; Barone-Adesi and Whaley, 1987). The Geske and Johnson (1984) gave an exact analytical solution for the American option pricing problem, but their formula is an infinite series that can only be evaluated approximately by numerical methods. The quadratic method (Barone-Adesi and Whaley, 1987; MacMillan, 1986) is based on exact solution to approximations of the option partial differential equation. The numerical solutions are expensive and do not offer the intuition which the comparative statics of an analytic solution provide. An analytic approximation has been developed by Johnson (1983), but it does not handle dividends or hedge ratio and there is no way to make this approximation arbitrarily accurate.

The aim of this paper is to determine the optimal critical price and to price the value of American call option under risk-neural measure, analytically. In this research, backward dynamic programming recursions (Winston, 1994; Dixit, 1990), the Martingale Pricing (Neftci, 2000), and the Girsanov Theorem (Neftci, 2000) will be applied to derive recursive formula. By using back forward method, the value of American call option on time t ($t < T$) is obtained and then the expected value of American call option on t is solved by Martingale pricing technique under risk neural measure to get recursive formula. Next section presents the analytical solution of non-dividend European call and dividend European call. In Section 3, a twice exercisable and triple exercisable call option and the fair value of calls and critical prices are derived. Furthermore, a multi-exercisable call option is evaluated and their critical price are obtained by mathematical induction.

2. BLACK-SCHOLES MODEL

Let S_T, σ^2, r, T, X, $C_T^E(X)$ and $C_n^A(X)$ be the underlying stock price on time T, the variance of the rate of return on the stock, the risk-free rate, the time to maturity of the call, the exercise price, the European call price with maturity T, and the American call option with n-th multi-exercisable call option with maturity T, respectively. Black and Scholes (1973) analysis which leads the fair value of an option is based mainly upon the following assumption. First of all, the perfect market, (2) the constant interest rate, r, and volatility, σ, and (3) geometric Brownian motion for the stock price. The current time is set to zero. The stochastic process for stock price change is assumed to be:

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{1}$$

where μ is the expected return on the stock and dz is the differential of a Gauss-Wiener process. Secondly, a hedge position is formed with a portfolio of short underlying and a long position of a number of European options. Then an arbitrage argument leads to the renowned Black-Scholes partial differential equation determining the value of the option.

$$\begin{aligned} C_T^E(X) &= e^{-rT} E[Max(S_T - X, 0)] \\ &= SN(d_1) - Xe^{-rT} N(d_2), \end{aligned} \tag{2}$$

where

$$d_1 = \frac{\ln(\frac{S}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

The function $N(x)$ is the cumulative probability distribution function for a standardized normal distribution. In other words, it is the probability that a variable with a standard normal distribution, $\varphi(0,1)$ will be less than x (Hull, 2003).

From Eq. (2), the Black-Scholes model exploiting the fact that a fair value of an option is given by the present value of the expected payoff at expiry under the risk-neutral measure. Then the payoff is computed and discounted up to the current time.

Merton (1973) valued the European option with a dividend yield at rate q.

$$\begin{aligned} C_T &= e^{-rT} E[Max(S_T - X, 0)] \\ &= e^{-qT} SN(d_1) - Xe^{-rT} N(d_2), \end{aligned} \tag{3}$$

where

$$d_1 = \frac{\ln(\frac{S}{X}) + (r + \frac{\sigma^2}{2} - q)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

3. VALUATION OF AMERICAN CALL OPTION

For valuation of American call, a procedure is used to price the American option under risk-neural measure and a twice exercisable call option is also used as a basis to extend to the multi-exercisable call option.

3.1 Valuation of Twice Exercisable Call Option

To see how to value a standard American call option, we start with the simple situation- a twice-exercisable option. That is we can exercise at a fixed time point, t_1 ($0 < t_1 < T$), before maturity or at maturity, *T*. Under the assumption of stock price following the geometric Brownian motion, the Black-Scholes model (1973) has been used to derive the European option expected payoff, that is $C_T^E(X) = e^{-rT}E\{Max[S_T - X, 0]\}$. If the American call option will not be exercised before expiration date which is just exercising at the end of period, the expected payoff of American option, $C_T^E(X)$, is the same as that of European option. Further problem is that under what condition, the holder will not exercise at the end of current period? If the stock price at time t_1, S_{t_1}, is lower than $X + e^{rt_1}C_T^E(X)$, the holder will not exercise on t_1. Because the payoff from exercising the call will be less than $C_T^E(X)$, under the risk-neutral measure, the holder will not exercise on t_1. And there will be $T - t_1$ left in the life of the call. In contract, if the stock price at time t_1 is higher than $X + e^{rt_1}C_T^E(X)$, the holder who will exercise right now, because the payoff from exercising the call will be more than $C_T^E(X)$. $X + e^{rt_1}C_T^E(X)$ is the middle price, M(*S*, *X*, *T*, t_1), at t_1. Therefore, the value of America call option, $V_{t_1}(X)$, is $Max\{S_{t_1} - X, e^{rt_1}C_T^E(X)\}$. The fair pricing of twice exercisable American call option, $C_2^A(X)$, is $e^{-rt_1}E[Max\{S_{t_1} - X, e^{rt_1}C_T^E(X)\}]$. By using Martingale Pricing and Girsanov Theorem (Neftci, 2000), the analytical solution of $e^{-rt_1}E[Max\{S_{t_1} - X, e^{rt_1}C_T^E(X)\}]$ is derived as follows.

$$C_2^A(X) = e^{-rt_1}E[Max\{S_{t_1} - X, e^{rt_1}C_T^E(X)\}]$$

$$= e^{-rt_1}E^Q[e^{rt_1}C_T^E(X)I_{\{0<S_{t_1}<K+e^{rt_1}C_T^E(X)\}}] + e^{-r}E^Q[(S_{t_1} - K)I_{\{S_{t_1}\geq K+e^{rt_1}C_T^E(X)\}}]$$

$$= e^{-rt_1}[1 - N(d_{2,t_1}(K + e^{rt_1}C_T^E(X)))]e^{rt_1}C_T^E(X)$$

$$+ e^{-qt_1}SN(d_{1,t_1}(K + e^{rt_1}C_T^E(X))) - e^{-rt_1}KN(d_{2,t_1}(K + e^{rt_1}C_T^E(X)))$$

$$= C_T^E(X) + e^{-qt_1} SN(d_{1,t_1}(K + e^{rt_1} C_T^E(X))) - e^{-rt_1}(K + e^{rt_1} C_T^E(X)) N(d_{2,t_1}(K + e^{rt_1} C_T^E(X)))$$

$$= C_T^E(X) + C_{t_1}^E(X + e^{rt_1} C_T^E(X)) . \tag{4}$$

where E^Q: The expected value of American option under Q measure in Girsanov Theorem,

I_A: An indicator variable, that is, $I_A = \begin{cases} 1, & \text{if sitution } A \text{ hold} \\ 0, & o/w, \end{cases}$

$$d_{1,t}(k) = \frac{\ln(\frac{S}{k}) + (r + \frac{\sigma^2}{2} - q)t}{\sigma\sqrt{t}} \text{, and} \tag{5}$$

$$d_{2,t}(k) = d_{1,t}(k) - \sigma\sqrt{t} . \tag{6}$$

If the call option has not already been exercised and the payoff from exercising the call option equals or exceeds $C_T^E(X) + C_{t_1}^E(X + e^{rt_1} C_T^E(X))$, it will not be exercised and the value of call option is $C_T^E(X) + C_{t_1}^E(X + e^{rt_1} C_T^E(X))$. Otherwise, the call option will be exercised and the value of call option is $S_0 - X$. This implies a "critical price" of twice exercisable call option is $X + C_T^E(X) + C_{t_1}^E(X + e^{rt_1} C_T^E(X))$. Figure 1 illustrates the fair value of twice exercisable call option, middle price and its critical price. In Figure 1, if the call option has not already been exercised before maturity, the value of American call option is 0 if the spot price is lower than exercise price, *X*. Otherwise, the value of American call is $S_T - X$ at maturity. When time is at t_1, the value of American call option is $e^{rt_1} C_T^E(X)$ if the spot price, S_{t_1}, is lower than $X + e^{rt_1} C_T^E(X)$. Otherwise, the value of American call option is $S_{t_1} - X$. Hence, $X + e^{rt_1} C_T^E(X)$ is a middle price at t_1 and the critical price of twice exercisable option is $X + C_T^E(X) + C_{t_1}^E(X + e^{rt_1} C_T^E(X))$.

3.2 Valuation of Triple Exercisable American Call Option

A triple exercisable call option is assumed the holder have three possible exercise epochs, at maturity (or *T*), t_1 and t_2 ($0 < t_1 < t_2 < T$). Similar to twice exercisable call option, the triple exercisable call option has two middle prices on t_1 and t_2. The middle price, M(*S*, *X*, *T*, t_2) on t_2 is $X + e^{rt_2} C_T^E(X)$ and also the middle price, M(*S*, *X*, *T*, t_1), on t_1 is $X + e^{rt_1}(C_T^E(X) + C_{t_2}^E(X + e^{rt_2} C_T^E(X)))$. Therefore, the value of America call option, $V_{t_1}(X)$, is $Max\{S_{t_1} - X, e^{rt_1}(C_T^E(X) + C_{t_2}^E(X + e^{rt_2} C_T^E(X)))\}$. The fair pricing of the American call

option, $C_3^A(X)$, is $e^{-rt_1}E[Max\{S_{t_1}-X, e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X)))\}]$. By using Martingale Pricing and Girsanov Theorem, the analytical solution of $e^{-rt_1}E[Max\{S_{t_1}-X, e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X)))\}]$ is derived as follows:

$$C_3^A = C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))+C_{t_1}^E(X+e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X)))) \quad (7)$$

Stock price / Time to maturity	Exercise price (X) Critical price: $X+C_T^E(X)+C_{t_1}^E(X+e^{rt_1}C_T^E(X))$) M(S, X, T, t_1)= $X+e^{rt_1}C_T^E(X)$			
End of period	**0,… ,0**	S_T-X , S_T-X ,…	S_T-X	S_T-X ,…
Time at t_1	$e^{rt_1}C_T^E(X)$,…	$e^{rt_1}C_T^E(X)$,…, $e^{rt_1}C_T^E(X)$	$S_{t_1}-X$	$S_{t_1}-X$
Now (Time period 0)	Do not exercise			Exercise

Figure 1 The Value of Twice Exercisable Call, Middle Price and its Critical Price

If the call option has not already been exercised and the payoff from exercising the call option equals or exceeds $C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))+C_{t_1}^E(X+e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))))$, it will not be exercised and the value of call option is $C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))$ $+C_{t_1}^E(X+e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))))$. Otherwise, the call option will be exercised and the value of call option is S_0-X . This implies a "critical price" of twice exercisable call option is $X+C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))+C_{t_1}^E(X+e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))))$. Figure 2 illustrates a fair value of triple exercisable call option, middle prices and its critical price. In Figure 2, there are two middle prices separately at t_1 and t_2. If the call option has not already been exercised before maturity, the value of American call option is 0 if the spot price is lower than exercise price, *X*. Otherwise, the value of American call option is S_T-X at maturity**.** When time is at t_2, the value

of American call option is $e^{rt_2}C_T^E(X)$ if the spot price, S_{t_2}, is lower than $X+e^{rt_2}C_T^E(X)$. Otherwise, the value of American call option is $S_{t_2}-X$. Similarly, when time is at t_1, the value of American call option is $e^{rt_1}(C_T^E(X)+C_{t_1}^E(X+e^{rt_1}C_T^E(X)))$ if the spot price, S_{t_1}, is lower than $X+e^{rt_1}(C_T^E(X)+C_{t_1}^E(X+e^{rt_1}C_T^E(X)))$. Otherwise, the value of American call option is $S_{t_1}-X$. Therefore, the critical price of twice exercisable option is $X+C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))$ $+C_{t_1}^E(X+e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))))$.

Stock price / Time to maturity	Exercise price (X)	M(S, X, T, t_2)= $X+e^{rt_1}(C_T^E(X)+C_{t_1}^E(X+e^{rt_1}C_T^E(X)))$ M(S, X, T, t_1)= $X+e^{rt_1}C_T^E(X)$			Critical price
End of period	**0,… ,0**	S_1-X **,** S_1-X **,…**	S_1-X	S_1-X **,…**	S_1-X
Time at t_2	$e^{rt_2}C_T^E(X)$ **,…**	$e^{rt_2}C_T^E(X)$ **,…,** $e^{rt_2}C_T^E(X)$	S_1-X	S_1-X	S_1-X
Time at t_1	$e^{rt_1}[C_T^E(X)+C_{t_1}^E(X+e^{rt_1}C_T^E(X))]$			S_1-X	S_1-X
Now (Time period 0)	Do not exercise				Exercise

Figure 2 The Value of Triple Exercisable Call, Middle Prices and its Critical Price

Tables 1 and 2 summarize the fair value of twice and triple exercisable call option and their critical prices. From Tables 1 and 2, the fair values of twice and triple exercisable call option, and their critical prices show three notable features. First of all, the relationship of both critical prices in twice and triple exercisable call options is not linear. This means that two types of exercisable call options have different critical price and their relationship is not linear.

Secondly, under the same assumption with Black-Sholes model (1973), the critical price is the

summation of European options' price with different exercise prices and different maturities add the exercise price.

Finally, the relationship of both the fair values or closed-form solutions in twice and triple exercisable call options is not linear.

Table 1 The Fair Value of Twice Exercisable Call Option and its Critical Price

The fair value of call option	$C_T^E(X)+C_{t_1}^E(X+e^{rt_1}C_T^E(X))$
Critical price	$X+C_T^E(X)+C_{t_1}^E(X+e^{rt_1}C_T^E(X))$

Table 2 The Fair Value of Triple Exercisable Call Option and its Critical Price

The fair value of call option	$C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))+C_{t_1}^E(X+e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))))$
Critical price	$X+C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))+C_{t_1}^E(X+e^{rt_1}(C_T^E(X)+C_{t_2}^E(X+e^{rt_2}C_T^E(X))))$

3.3 Valuation of Multi-exercisable Call Option

A general situation, with n possible exercised epochs at time t_1, $t_2,\ldots t_{n-1}$ and maturity, T, ($0<t_1<t_2<\ldots<t_{n-1}<T$) is considered. Based on the discussion in section 3.1 and 3.2, there are (n-1) middle prices and C_n^A is obtained from Eqs. (4) and (7).

$$C_n^A = C_T^E(X)+C_{t_{n-1}}^E(X+e^{rt_{n-1}}C_T^E(X))$$

$$+C_{t_{n-2}}^E(X+e^{rt_{n-2}}[C_T^E(X)+C_{t_{n-1}}^E(X+e^{rt_{n-1}}C_T^E(X))])+\ldots.+C_{t_1}^E(X+\ldots) \quad (8)$$

Therefore, the critical price of American call option, $\overline{S}$,is

$$\overline{S}=X+C_T^E(X)+C_{t_{n-1}}^E(X+e^{rt_{n-1}}C_T^E(X))$$

$$+C_{t_{n-2}}^E(X+e^{rt_{n-2}}[C_T^E(X)+C_{t_{n-1}}^E(X+e^{rt_{n-1}}C_T^E(X))])+\ldots.+C_{t_1}^E(X+\ldots) \quad (9)$$

Eqs. (8) and (9) have three features. First of all, the analytical solution is the summation of n European call options' prices with different maturities and different exercise prices. There are two advantages of them: (1) the valuation of American call option can be obtained easily by the summation of Black-Sholes formula (or European call option pricing formula), and (2) an analytical formula is more efficient than approximating the stock price process or the partial differential equation by binomial

(Cox et al., 1979) or finite difference methods (Brennan and Schwartz, 1978).

Secondly, from Eqs. (8), the same conclusion is compared with Barone-Adesi and Whaley's (1987) conclusion; the value of American call option is the value of European call option add early exercise premium. Thus, the early exercise premium, $\varepsilon_C(S,T)$, can be obtained easily. And $\varepsilon_C(S,T)$ can be expressed mathematically as

$$\varepsilon_C(S,T) = C_n^A(X) - C_T^E(X) \tag{10}$$

Thirdly, from Eqs. (8) and (10), the features of the early exercise premium, $\varepsilon_C(S,T)$, can be easily captured by the Greek Letters' property (Hull, 2003).

4. CONCLUSIONS

The valuation of American options on dividend-paying asset is an important problem in financial economics. The theoretical values of European option can be evaluated by a simple formula. However, the theoretical values of American options are very difficult to determine since the possibility of American options can be early exercise. In this study, analytical solution of pricing American call option and the critical price are obtained. And from analytical solution and the critical price can get two features. First of all, the analytical solution is the summation of *n* European call options' prices with different maturities and different exercise prices. There are two advantages of them: (1) the valuation of American call option can be obtained easily by the summation of Black-Sholes formula (or European call option pricing formula), and (2) an analytical formula is more efficient than approximating the stock price process or the partial differential equation by binomial (Cox et al., 1979) or finite difference methods (Brennan and Schwartz, 1978). Secondly, the same conclusion is compared with Barone-Adesi and Whaley's (1987) conclusion; the value of American call option is the value of European call option add early exercise premium. Previous attempts at pricing these options have been accurate but computationally expensive. This study provides a simple, an inexpensive solution for pricing American call option.

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