Measuring Bond Portfolio Value At Risk: Us And Taiwan Government Bond Markets Empirical Research

Thomas W. Knowles
Stuart Graduate School of Business
Illinois Institute of Technology, USA
knowles@stuart.iit.edu

Ender Su
National Kaohsiung First University of Science and Technology, Taiwan
suender@ccms.nkfust.edu.tw

Abstract

This paper concerns value at risk analysis of US and Taiwan government bond portfolios. We explore four methods of modeling the yield curve and its risk: key rate, three-factor model (level, slope, and curvature), principal component analysis (PCA), and structural equation modeling (SEM). SEM has never been used previously for analyzing bond risk. We found that the data analysis methods obtained similar VaR figures for the US and Taiwan. For a one-day investment horizon, using the key rate the values were $2.511 and $2.507, respectively, and using PCA the values were $2.5104 and $2.506. The model fit methods found smaller VaR figure: using the three-factor model the VaR was $2.4642 and $2.4902, for the US and Taiwan, respectively, and $2.4941 and $2.4918, using SEM. The key rate, PCA and SEM all suggest that US has a slightly higher VaR value than Taiwan. For US bond yield risk, the slope of the three-factor model and the medium term of the SEM exhibit higher VaR sensitivity while for Taiwan, the level of the three-factor model and the medium term of the SEM model risk factors present higher VaR sensitivity. The three-factor model and SEM provide understandable method for interest rate risk management and forecasting, but because they tend to underestimate the risk, data analysis methods such as key rate and PCA are needed to monitor bond yield curve risk.

1. Introduction

The value of a bond is subject to movement of the yield curve. Many researchers such as Bloomberg, Standard & Poor, Morgan Stanley, and JP Morgan display the US yield curve chart movement every day on their web sites. We can observe from those sources how the yield curve flows with changes in the US economy. Notably, the yield curve flows dramatically when the Federal Reserve Bank announces an interest rate change. Typically, the yield curve will be upflow when the US economy is strong and downflow when the US economy is weak. On the other hand, Taiwan has become a WTO member, and we expect that the Taiwan government bond market will be traded actively and liberally. Thus, certainly studying the yield curve of the Taiwan bond market is important and necessary for the growth of the Taiwan bond market risk measure. It will be interesting to compare the US and Taiwan bond market risks with respect to yield curve model factors and economic factors. We will apply four models: key rate, three factors--level, slope, and curvature, principal component analysis (PCA), and structural equation modeling (SEM) to analyze and detail the yield curve movement and its risk for the US and Taiwan government bond markets.

With respect to interest rate risk and value at risk (VaR) empirical study, J P Morgan in its 1996 year "technical document" [6] has a detailed description for the key rate variance-covariance value at risk measurement. In reality, key rates play the important role for the standardized risk factors. As to the individual bond or bond portfolio, we can just use J P Morgan's provided key rate parameters (variance-covariance matrix and key rates), and obtain the bond value at risk easily. If the bond cash flow periods don't match the key rate periods used in RiskMetrics, then by the cash flow distribution methods, e.g. duration mapping or variance equality, we distribute the unmatched bond cash flow to the key rate vertices, i.e. the standardized key rates. Similarly, we can use the key rate parameters provided by J P Morgan to acquire the linear value at risk. In addition, key rate duration (Ho [4]) also can be applied to the cash flow mapping and bond value at risk measure. Golub and Tillman [4] have applied this method for the derivation of the bond value at risk formulation. [4] has derived the key rate "factor block" duration that can be used for the VaR measure and VaR components of a bond portfolio.

RiskMetrics or key rate duration both retain the problem of dependence between different key rates. In other words, how the relationships of key rate factors capture the real world yield curve movement is the key point for the key rate application of bond value at risk. However, the key rate factor dependence will become an obstacle for investors since they must consider how well the key rate factor relationships match the real yield curve movement.

Principal component analysis (PCA) is another data analysis model like the key rate method. Like the three-factor model, principle component analysis reduces the numerous yield change components to a few major components and thus reduces the number of parameter estimates for the measure of the bond value at risk. A few major component variances can explain almost 99% system variance and most importantly the components are independent of each other. Singh [9] compared the performances of the key rate duration, the three-factors yield curve model, and offered principal component analysis of the yield curve risk. Golub and Tillman [4] also discussed using the principal component durations to measure the bond value at risk.

A new method of modeling the yield curve is structural equation modeling (SEM), which encompasses many research subjects such as covariance structure analysis, latent variable analysis, and exploratory factor analysis. SEM has not been used previously to model the yield curve. With the help of LISREL and Amos, SEM has been applied in other contexts (e.g. Austin & Calderon [1], Tremblay & Gardner [10]). In addition to the PCA exploratory factor analysis, we also will adopt the SEM confirmatory factor analysis that will search for the parsimonious unobserved latent factors to explain the observed known endogenous variables' variances. Unlike PCA that requires that the factor components should be orthogonal, the SEM confirmatory factor will allow some relationships among the latent factors.

Wilson [14] presented the delta gamma approximation for interest rate risk measures of nonlinear derivative securities. Also, JP Morgan RiskMetrics has offered users variance-covariance matrix and the delta gamma method to calculate the interest rate portfolio risk measure. More recently, Khindanova and Rachev [7] and Bodurtha and Shen [2] give portfolio value at risk measures of bond options and foreign currencies. The major risk measure process is to distribute the cash flow amount into delta, gamma, and theta cash flows and then use the variance-covariance matrix of their cash flows to measure the portfolio bond value at risk. Vlaar [12] and Venkatesh [11] proposed a GARCH type model to analyze the time series properties of bond portfolios and test their VaR estimates.

2. Research Methodology

We prefer using the factor involved models to explain the movement of the yield curve and the value at risk measures of the risk factors (components), rather than use time series models such as GARCH type models to forecast the bond yield volatilities and forgo the risk factor measures. Therefore, we use four linear models for our bond VaR measures: key rate duration, three-factor model-level, slope, and curvature-, principle component analysis, and structure equation modeling. After modeling the factor's coefficients, we can calibrate VaR and factor component VaR through the factor's coefficient transformation to give the risk measure. Because we find all the model parameters, we also will be able to compare each model's description of the yield curve behavior. While each model will explain the yield curve in different respects, we will find the advantages and disadvantages of each method of yield curve modeling from different aspects, such as the model factor representations, sensitivities, correlation, variance covariance, and the overall model significance tests.

2.1 Bond Portfolio VaR Measure Model I: Key Rate Duration Structure and its VaR

TheRiskMetrics technical document describes the construction of the key rate duration structure. First, it says the yield curve would be divided into several knots or vertices according to preselected maturities, months or years, named as key rates. Then, the actual cash flows of different maturity bonds are distributed into the preselected key rates. There are two methods for distributing the actual cash flow into the preselected key rates. One is duration mapping, and the other is cash flow mapping. We will explain these later.

When c_i is the distributed key rate cash flow, r_i is the i^{th} key rate (i=1, 2…n), and V is the total portfolio position, the key rate duration can be defined as follows:

$$k(i) = -(dv_i / dr_i)/V$$
$$= c_i / [(1+r_1)...(1+r_i)] * MDuri / V$$
$$= v_i / V * MDuri \qquad (1)$$

where $MDuri$ is the ith modified duration and v_i is the i^{th} distributed cash flow key rate. As to the VaR of a bond portfolio, we calculate the portfolio VaR function (at a 95% confidence level) as

$$VaR = 1.65\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} v^i \sigma_i^p v^j \sigma_j^p corr(i,j)}\,, \qquad (2)$$

where v_i and v_j are the i^{th} and j^{th} key rate cash flow position, and σ_i^p, σ_j^p and $corr(i,j)$ are the i^{th} and j^{th} key rate cash flow variance and correlation. (Note: The p refers to price and is not an exponent.) And, the first derivative of the bond price (P) function (proposed by Fisher[3]) is

$$dP = -MDur * P * dy\,. \qquad (3)$$

Taking the variance of the bond side equation, we obtain

$$\sigma^p = MDur * y * \sigma^y\,. \qquad (4)$$

And, the VaR is

$$VaR = 1.65V\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} k_i y_i \sigma_i^y k_j y_j \sigma_j^y corr(i,j)} \qquad (5)$$

$$VaR = 1.65V\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} k_i k_j \operatorname{cov}(\Delta y_i, \Delta y_j)}\,. \qquad (6)$$

We know $\frac{\Delta V}{V} = -\sum_{i=1}^{n} k_i \Delta y_i$ and if we rewrite the above equation in terms of matrix expression, we can find

$$VaR = 1.65V\sqrt{k\Omega k'}\,, \qquad (7)$$

where Ω is the variance-covariance matrix of the yield change.

RiskMetrics uses cash flow mapping for cash flows of maturities not on the vertices. Under the principle of volatility equality, the actual cash flows are distributed into the preselected key rates. The procedure is to find the parameters $\hat{a}$ and w_i that keep the volatility equal after mapping using the following equations:

$$\hat{\sigma}_{vertic} = \hat{a}\sigma_i + (1-\hat{a})\sigma_j \qquad 0 \le a \le 1 \tag{8}$$

$$\hat{\sigma}^2_{vertic} = w_i^2\sigma_i^2 + 2w_i(1-w_i)\rho_{i,j}\sigma_i\sigma_j + (1-w_i)^2\sigma_j^2, \tag{9}$$

where $\hat{a}$ is the maturity weight of volatility that can be estimated first by linear interpolation by maturity between ith and jth cash flows (so, if the maturity is in the middle of the maturities, it is estimated as 0.5). w_i is the weight of cash flow mapping that can be solved for secondly by equation (9) with only w_i unknown . Thus, if we can estimate parameter $\hat{a}$ and solve for parameter w_i, then we can use above key rate duration to measure bond cash flow VaR.

2.2 Bond Portfolio VaR Measure Model II: Level Slope Curvature Yield Curve Modeling and its VaR

Nelson and Siegel [8] derived the level (L), slope (S), and curvature (C) yield curve model and Willner [13] further applied this model. The yield curve model is used to explain the effects on the bond yield curve due to the economic factor changes such as inflation, the business cycle, and interest rate volatility changes. We derive the model as follows:

$$Y(L,S,C,m) = L + (S+C)\frac{(1-e^{-m/\tau})}{m/\tau} - Ce^{-m\tau}, \tag{10}$$

where m is the maturity and 塲is the location parameter (hump or vertex). Further, we can define $X_1(m) = (1-X_2(m))/(-m/\tau)$ and $X_2(m) = e^{-m/\tau}$; thus the yield curve function becomes

$$Y(L,S,C,m) = L + X_1(m)S + (X_1(m) - X_2(m))C \tag{11}$$

We obtain the level, slope, and curvature durations by taking the derivative of the yield curve function to obtain

$$dY = \frac{dY}{dL}\Delta L + \frac{dY}{dS}\Delta S + \frac{dY}{dC}\Delta C \tag{12}$$

$$\frac{dY}{dL} = 1,\ \frac{dY}{dS} = X_1(m),\ \frac{dY}{dC} = (X_1(m) - X_2(m)). \tag{13}$$

If the bond value is

$$B(m) = \Sigma\frac{c_t}{(1+Y(m))^t}, \tag{14}$$

(recall c_t is the t th bond cash flow), the bond price changes can be described as

$$\begin{aligned} dB &= \frac{dB}{dY}dY \\ &= \frac{dB}{dY}\left(\frac{dY}{dL}\Delta L + \frac{dY}{dS}\Delta S + \frac{dY}{dC}\Delta C\right) \\ &= \frac{dB}{dY}(dL + X_1(m)dS + (X_1(m) - X_2(m))dC). \end{aligned} \tag{15}$$

And, through the known three-factor coefficients, we can derive the VaR as the following equation shown:

$$VaR = 1.65\sqrt{\sum_{m=1}^{3}\sum_{n=1}^{3} v_m\sigma_m v_n\sigma_n corr(m,n)}, \tag{16}$$

where v_m and v_n are the cash flows attributed to the three factors.

2.3 Bond Portfolio VaR Measure Model III: Principle Component Analysis and its VaR

Principle component analysis is also a linear structure model that tries to find the major linear factors (usually three) to explain the variances of the research objective dependent variables (often more than three). It is quite interesting to compare those linear models and find valuable consequences among those models such as the factor volatility explanation ability, the factor relationships, and the factor representations. We can obtain the eigen vectors and values needed to obtain the principle component factors by maximizing the variance of the PCA factors after transforming the variance-covariance of dependent variables. Suppose that the key rate variance-covariance matrix is Σ, the eigen vector is c, and the eigen value is λ. Then,

$$\Sigma c = \lambda c, \tag{17}$$

The above equation expressed in terms of matrix operations is

$$\Sigma\begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \cdots & \cdots & \cdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \cdots & \cdots & \cdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \cdots & \\ & & \lambda_n \end{bmatrix}, \tag{18}$$

$$\Sigma C = C\Lambda, \tag{19}$$

where C is called the characteristic matrix, and is orthogonal to itself; thus

$$CC' = I, \tag{20}$$

so that

$$C'\Sigma C = C'C\Lambda = \Lambda. \tag{21}$$

Due to the principle component factor independence, we thus could find the 95% confidence level would be

$$VaR = 1.65\sqrt{\sum_{i=1}^{n} v_i^2\sigma_i^f}, \tag{22}$$

where v_i is the i^{th} principle component cash flow with respect to the one basis point change, σ_i^f is the i^{th} variance of the principle component, i.e. the characteristic matrix. The v_i can be found through the characteristic matrix transformation as the following:

$$v_i = \sum_j p_j c_{ij}, \tag{23}$$

where c_{ij} is the j^{th} element of the i^{th} characteristic component vector matrix and p_j is the present value of the j^{th} cash flow.

2.4 Bond Portfolio VaR Measure Model IV: Structure Equation Model

We adopt the SEM confirmatory factor model. If there are n set of measures with m_1, $m_2, \ldots,$ m_n ($q=m_1+m_2+\ldots+m_n$) variables in each set respectively, then we can set

$$X = \Lambda\eta + \varepsilon, \tag{24}$$

where X is the different maturity yield data vector with order of q by one, η is the structure factor n by one vector, Λ is the structural coefficient matrix of X on η with order of q by n, and ε is the error term with order of q by one. We note that confirmatory factor analysis needs to be done to find the appropriate factor connections with the observable variables.

3. Yield Curve Model Applications For US and Taiwan Government Bonds

The US government bond market is the most efficient interest rate market in the world. Many financial institutions-banks, insurance, and investors- make a large number of interest rate transactions to invest, hedge, speculate or arbitrage interest rate changes. US treasury bills, treasury notes and treasury bonds are the most traded securities and have different maturities. By issuing them, the US government can finance its budget deficit or refinance its old debt. Since 1991 the Taiwan government bond market has adopted bid-ask price auctions, and over time the bond market in Taiwan will become more liquid and efficient. The trading volume has increased since 1991.

3.1 Research Data

The US Treasury bond yield curve data can be acquired from the web database of the US Federal Reserve Bank. However, the yield data of different maturity Taiwan government bonds would be difficult to obtain since the Taiwan bond market is not as active and public. We can find the bond data from Aremos databank or the R.O.C. Over The Counter Security Exchange databank. For the US government bond yield data, we use 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, 10-year, 20-year, and 30-year interest rates because those maturities matched Riskkmetrics' key rate maturities. We used maturities of 3-month, 6-month, 1-year, 3-year, 5-year, 7-year, and 10-year for the Taiwan government bond yield data. However, Taiwan's long-term bonds have been traded sporadically. There is not enough data in Aremos and it has questionable integrity. Thus, we must interpolate Taiwan's long-term data for 5-year, 7-year, and 10-year maturity bond data. We primarily use spline interpolation methods to form the time series of its bond data. Unfortunately, there is little data for longer terms such as 15 years and more; thus we forgo terms longer than 15 years for Taiwan. The research periods for US bond yield data are from 1/14/1999 to 1/14/2005 and from 21/04/1999 to 1/14/2005 for Taiwan bond yield data. This yields a total sample size of 1502 observations for each country's bond yield data.

3.2 Key Rate Duration Value at Risk

First, we will use the key rate duration method to measure the bond value at risk. Due to the wide range of different bond maturities, we use RiskMetrics' key rates and use maturities of 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years, and 30 years as our research key rates. Furthermore, JP Morgan RiskMetrics has provided its US bond market index. Thus, we will use this index as the base investment amount for the measure of bond value at risk. In Table 1, we present the key rate correlation/variance-covariance matrix for the computation of the VaRs for US and Taiwan bond yield data.

Table 1 Key Rate Correlation/Variance-Covariance Matix (page 1 of 2)

I. US Bond Yields

Maturity	Spot Rate	Stdev	CM3m	CM6m	CM1Y	CM2Y	CM3Y	CM5Y	CM7Y	CM10Y	CM20Y	CM30Y
CM3m	2.370%	1.929%	3.72E-04	3.56E-04	3.52E-04	3.25E-04	2.86E-04	2.18E-04	1.85E-04	1.37E-04	9.18E-05	1.47E-04
CM6m	2.610%	1.853%	0.998	3.43E-04	3.40E-04	3.15E-04	2.78E-04	2.13E-04	1.81E-04	1.34E-04	9.00E-05	1.42E-04
CM1Y	2.870%	1.844%	0.990	0.996	3.40E-04	3.18E-04	2.82E-04	2.17E-04	1.85E-04	1.38E-04	9.34E-05	1.44E-04
CM2Y	3.240%	1.740%	0.968	0.979	0.992	3.02E-04	2.70E-04	2.11E-04	1.81E-04	1.36E-04	9.31E-05	1.40E-04
CM3Y	3.410%	1.558%	0.951	0.964	0.981	0.997	2.43E-04	1.91E-04	1.64E-04	1.24E-04	8.56E-05	1.27E-04
CM5Y	3.710%	1.232%	0.920	0.934	0.956	0.984	0.994	1.52E-04	1.31E-04	1.00E-04	7.00E-05	1.02E-04
CM7Y	3.970%	1.068%	0.900	0.914	0.940	0.972	0.985	0.997	1.14E-04	8.74E-05	6.18E-05	8.91E-05
CM10Y	4.230%	0.823%	0.861	0.877	0.907	0.947	0.965	0.987	0.993921	6.78E-05	4.82E-05	6.81E-05
CM20Y	4.760%	0.596%	0.799	0.815	0.850	0.899	0.923	0.954	0.972	0.983	3.55E-05	4.83E-05
CM30Y	4.640%	0.867%	0.878	0.885	0.904	0.932	0.944	0.959	0.963	0.954	0.935	7.51E-05

Note: the correlation is in the lower triangle and the variance covariance is in the upper triangle.

II. Taiwan Bond Yields

Maturity	Spot Rate	Stdev	CM3m	CM6m	CM1Y	CM3Y	CM5Y	CM7Y	CM10Y
CM3m	0.996%	1.66%	2.7E-04	2.8E-04	2.5E-04	2.62E-04	2.3E-04	2.2E-04	2.3E-04
CM6m	0.972%	1.69%	0.989	2.8E-04	2.5E-04	2.62E-04	2.3E-04	2.2E-04	2.3E-04
CM1Y	1.698%	1.54%	0.995	0.979	2.4E-04	2.45E-04	2.2E-04	2.1E-04	2.2E-04
CM3Y	3.128%	1.62%	0.978	0.960	0.982	2.62E-04	2.3E-04	2.2E-04	2.3E-04
CM5Y	3.323%	1.47%	0.948	0.929	0.953	0.967	2.2E-04	2.0E-04	2.1E-04
CM7Y	4.487%	1.48%	0.900	0.885	0.915	0.920	0.918	2.2E-04	2.0E-04
CM10Y	3.096%	1.49%	0.932	0.909	0.940	0.961	0.960	0.916	2.2E-04

Note: the correlation is in the lower triangle and the variance covariance is in the upper triangle.

It shows there are standard deviations greater than 1% from 3-month to 7-year maturities in US bond data and higher correlations between adjacent maturities (the values just below the diagonal) in both US and Taiwan bond data. Comparing the countries' bond data, shorter-term interest rates of US bonds have higher volatilities whereas longer-term interest rates of Taiwan bonds have higher volatilities. On the other hand, the bond index (i.e. the supposed investment positions among different maturities) doesn't need to be mapped into key rates since we use JP Morgan's interest rate index structure to analyze our key rate duration and value at risk. Therefore, with the key rate variance-covariance and VaR(%) on hand and according to the bond index investment, we find our bond portfolio value at risk.

Table 2 contains the value at risk figures of US and Taiwan bond yields based upon the JP Morgan bond index. Suppose that our bond index investment is $100 (in $million); we estimate the one-day VaR according to equation (2) in the US and Taiwan government bond markets as about $2.511 and $2.507 (in $million), respectively, for a one-day investment time horizon. We see that the US bond yield has bond VaR amounts greater than Taiwanese bonds even considering the differences between the duration, the key rate volatility, and the correlation of the key rates. Though, we should note that the second order effect of the bond yield- convexity- would affect bond risk measure as well. Notably, in the US bond market, the diversification effect on the VaRs show the bond portfolio don't have lower risk even though the bond maturity spread becomes larger or when more of the bonds with different maturities are involved in the bond investment. We see the bond index position has a VaR of $2.511, greater than that for positions 1, 2 and 3. But, position 4, consisting of only two maturities (3 month and 30 year) with a larger maturity difference, has a larger VaR, $2.928. This indicates that the convexity and diversification effects are very small among US bond yield data. On the contrary, in Taiwan, the bond index position has a lower VaR value than position 1 and a higher VaR value than position 2, which indicates that convexity and diversification effects exist in Taiwan bond yield data.

Table 2 JP Morgan Bond Index Investment Value at Risk (page 1 of 2)
(daily base, 95% confidence level)

(1) US Government Bonds

Maturity	Spot Rate	VaR(%)	Bond Index Weights	Key Rate Duration	position1	position2	position3	position4
CM3m	2.37%	3.172%	$1.73	0.0043291				0.871933
CM6m	2.61%	3.048%	$3.19	0.0159697				
CM1Y	2.87%	3.033%	$17.91	0.1790662			0.66	
CM2Y	3.24%	2.862%	$31.85	0.6369933		0.588		
CM3Y	3.41%	2.563%	$19.75	0.5926116	0.47			
CM5Y	3.71%	2.026%	$9.77	0.4887122	0.53			
CM7Y	3.97%	1.757%	$5.79	0.4049513		0.412		
CM10Y	4.23%	1.355%	$4.28	0.4284248			0.34	
CM20Y	4.76%	0.980%	$4.04	0.8081067				
CM30Y	4.64%	1.426%	$1.68	0.5041047				0.128067
			Total = $100.00					
			Duration=	4.06	4.06	4.06	4.06	4.06
			Undiversified VaR=	$2.534	$2.279	$2.407	$2.462	$2.949
			VaR=	**$2.511**	$2.275	$2.393	$2.427	$2.928
			Diversification Effect=	$0.023	$0.004	$0.014	$0.035	$0.021
			Diversification Effect (%)=	0.92%	0.16%	0.58%	1.43%	0.71%

Table 2 Continued (page 2 of 2)(2) Taiwan Government Bonds

Maturity	Spot Rate	VaR(%)	Bond Index Weights	Key Rate Duration	Position1	Position2	Position3	Position4
CM3m	0.996%	1.66%	$1.92	0.00481			0.4770369	0.6379
CM6m	0.972%	1.69%	$3.55	0.01775		0.2711111		
CM1Y	1.698%	1.54%	$19.90	0.19897	0			
CM3Y	3.128%	1.62%	$35.39	1.06172	1			
CM5Y	3.323%	1.47%	$21.95	1.09749		0.7288889		
CM7Y	4.487%	1.48%	$10.86	0.76026			0.5229631	
CM10Y	3.096%	1.49%	$6.43	0.64282				0.3621
			Total = $100.00					
			Duration=	3.78	3.00	3.78	3.78	3.78
			Undiversified VaR=	$2.553	$2.663	$2.518	$2.577	$2.629
			VaR=	**$2.507**	$2.663	$2.481	$2.512	$2.589
			Diversification Effect=	$0.046	$0.000	$0.038	$0.065	$0.040
			Diversification Effect (%)=	1.79%	0.00%	1.49%	2.54%	1.53%

What if the bond cash flow cannot match our key rate maturity? Then, a cash flow mapping will be needed, using either variance equality or duration equality. Consider an example portfolio consisting of three kinds of US bonds: 3% coupon 1-year bonds, 5% 3-year bonds, and 6% 5-year bonds. We invest a total amount of 317.39 millions in the portfolio. According to the cash flow mapping (by principle of equality variances), we can obtain cash flow allocation as shown in Table 3-1. The weight of the interest rate volatility is the base for the cash flow distribution between 3-year and 5-year key rates. Therefore 50.68% of the 4-year cash flow will be allocated into the 3-year key rate and 49.32% into the 5-year key rate. Finally, the VaR of the 1, 3, and 5-year bond portfolio is computed as $8.055 or 2.315%, which is less than JP Morgan's bond index VaR of 2.511%, because of the lower volatilities and correlations.

Table 3-1 Bond Portfolio Cash Flow Distribution on the Key Rates by Variance Equality and its Value at Risk

Maturity	Spot Rate	VaR(%)	1yr	3yr	5yr	Cash Flow	Present Cash Flow	Key Rate Present Cash Flow	CashVaR	Correlation			
CM3m	2.37%	3.172%											
CM6m	2.61%	3.048%											
CM1Y	2.87%	3.033%	$103	$5	$6	$114	$110.82	$110.82	$3.36	1.0000	0.9624	0.9171	0.956
CM2Y	3.24%	2.862%		$5	$6	$11	$10.36	$10.36	$0.30	0.9624	1.0000	0.9887	0.984
CM3Y	3.41%	2.563%		$105	$6	$111	$101.07	$104.02	$2.67	0.9171	0.9887	1.0000	0.994
CM4Y	3.71%	2.026%			$6	$6	$5.28	$0.00	$0.00				
CM5Y	3.97%	1.757%			$106	$106	$89.87	$92.20	$1.87	0.956	0.984	0.994	1
CM7Y	4.23%	1.355%			Total=	$348	$317.39	$317.39	$8.19				
CM10Y	4.76%	0.980%							Vol Weitght =	50.68%			
CM20Y	4.64%	1.426%							Undiversified VaR=	$8.1914			
CM30Y	2.37%	3.172%							VaR =	$8.0555	%VaR=	2.315%	
									Diversified Effect=	$0.1358	(1.658%)		

Table 3-2 Key Rate Cash Flow Difference Effect upon Value at Risk

Cash Flow Difference Between 1 and 5 yr Key Rates	$1.00	$3.00	$5.00	$7.00	$9.00	$11.00	$13.00	$15.00
VaR	$8.07	$8.09	$8.14	$8.21	$8.30	$8.42	$8.57	$8.73
%VaR	2.318%	2.325%	2.338%	2.359%	2.386%	2.420%	2.461%	2.510%
Diverfied VaR	$0.0166	$0.0167	$0.0169	$0.0171	$0.0173	$0.0176	$0.0177	$0.0178
Key Rate Cash Flows CM1Y	$115.00	$117.00	$121.00	$127.00	$135.00	$145.00	$157.00	$171.00
CM2Y	$11.00	$11.00	$11.00	$11.00	$11.00	$11.00	$11.00	$11.00
CM3Y	$111.00	$111.00	$111.00	$111.00	$111.00	$111.00	$111.00	$111.00
CM4Y	$6.00	$6.00	$6.00	$6.00	$6.00	$6.00	$6.00	$6.00
CM5Y	$105.00	$103.00	$99.00	$93.00	$85.00	$75.00	$63.00	$49.00
Total Cash Flow	$348.00	$348.00	$348.00	$348.00	$348.00	$348.00	$348.00	$348.00
Cash Flow Difference	$10.00	$14.00	$22.00	$34.00	$50.00	$70.00	$94.00	$122.00

Table 3-2 shows the cash position difference effect upon the bond portfolio VaR number. If there is more weight on cash investment in the 3-month maturity bond, the VaR figure of the bond portfolio will increase more rapidly because of the high volatility of the 3-month bond yield and the low volatility of the 5-year bond yield.

3.3 Level, Slope, and Curvature Duration Value at Risk

We first review the yield curve term structure change for both countries' bond data. Figure 1 presents the US bond yield curve change from early 1999 to early 2005 according to equation (10). It shows the level, slope, and curvature movements of the yield curve as time passes. We find that in mid-1999, US interest rate structure exhibited a higher level for the term structure, and relatively lower structural slope and curvature. There is some evidence showing that from early 1999 to late 2001, the US seemed to have modest economic growth. The US Federal Reserve raised its discount interest rate hoping to cool the economy. Contrary to mid-1999, we find that in mid-2003, the US interest rate structure exhibited a tendency to a lower level and a higher slope. In early 2005, the US interest rate structure has a lower level, slope and curvature. The term structure movement tells us since early 2002, the US economy is gradually weakening. Especially, the higher slope in the previous period has dropped and become nearly flat at present. We consider the level of the interest rate term structure as reflective of the inflation situation, the slope of the interest rate term structure reflective of the economic

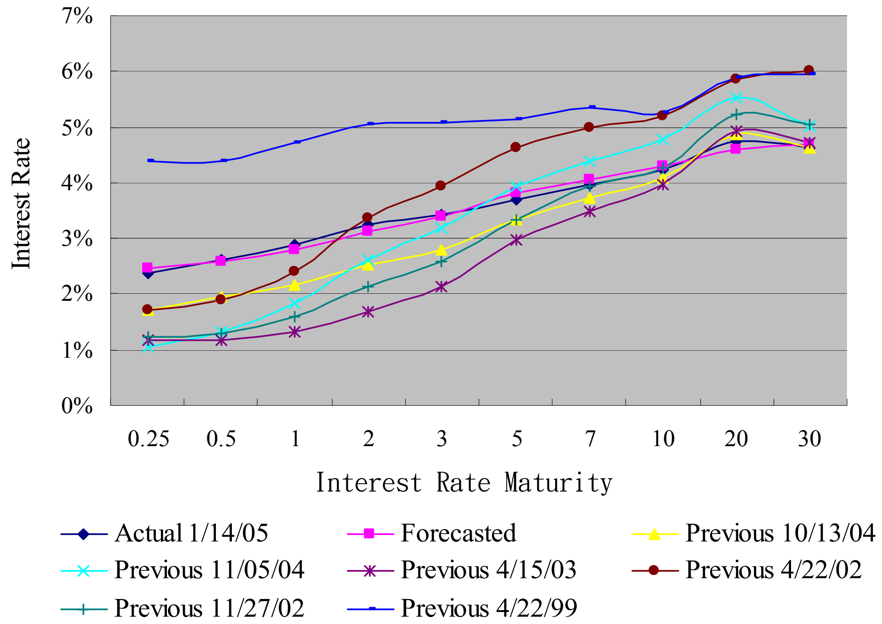

Figure 1 US Bond Yield Term Structure Movement

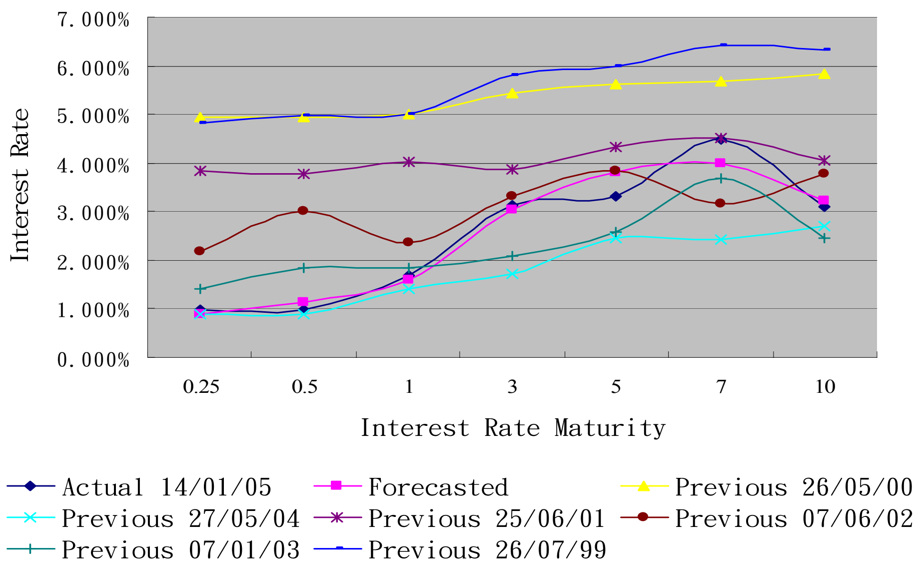

Figure 2 Taiwan Bond Yield Term Structure Movement

business cycle, and the curvature of the interest rate term structure reflective of economic volatility. Clearly in the US term structure movement, the level and slope have a positive relationship, i.e. when the level is high, the economy becomes stronger, and when the level is high, the slope will likely also have a high level, and the economy is going to get stronger.

On the other hand, in Taiwan, recently short-term interest rates have came down more than in the US. Investors such as banks, insurers and other financial institutes anticipate that the economy will be stagnant for some future years, as people in Taiwan prefer to invest and do business in foreign countries because of their cheaper labor and product material. Figure 2 displays the slightly negative relationship between level and slope for Taiwan bond data. Recently, in Taiwan, the economy is becoming weaker and the level drops deeply with the longer term interest rate going downward slightly and thus the slope becomes higher, not lower.

This might be explained because as the economy grows, people will prefer longer term borrowing and short term investment since people predict a lower short rate and the higher long rate will go down, and as the economy weakens, people do the opposite in the US bond market. In Taiwan, people will predict an increasing long-term rate as the economy grows and react differently than people in the US.

Next, to measure the bond value at risk by the three-factor yield curve model, we have estimated the three factors' parameters. Table 4 provides the level, slope, and curvature correlations, and variance covariance information computed by using factor transformations with the whole period yield curve data. As to the bond value at risk measure, we must find the duration to specify the level, the slope, and the curvature, like the key rate duration. Since the level is fixed at one, we can clearly recognize it has the duration equal to our cash flow. The slope and the curvature formed by our three-factor model would need to be estimated and optimized according to equation (10). Using the least squares forecasting error method, we obtain Table 5 that shows both the US and the Taiwan estimates of the three-factor model parameters. We obtain the optimal tau (τ) by minimizing the sum of the squared errors where the forecasts come from equation (10). The τ is our optimal term structure vertex (hump), the level remains one, and the slope and the curvature are estimated by τ. For the US interest rate term structure τ is estimated as around 1.786 years and for Taiwan it is estimated around 50.078 years as shown in Table 5. The longer term of the vertex in Taiwan partly results from the data's shorter interest rate term structure and partly from its higher volatility and curvature components. We used the three-factor yield curve model to approximate the yield curve term structure of 01/15/2005 and we then obtain a term structure yield curve forecast as shown in Figure 1 for US bond yield data and in Figure 2 for Taiwan bond yield data.

Table 4 Level Slope Curvature Correlations

(1) US Bond Yields

Correlation	Level	Slope	Curvature
Level	1.000		
Slope	0.555	1.000	
Curvature	0.620	0.381	1.000
Variance-Covariance	Level	Slope	Curvature
Level	2.659E-05		
Slope	4.787E-05	2.788E-04	
Curvature	5.451E-05	1.080E-04	2.888E-04

(2) Taiwan Bond Yields

Correlation	Level	Slope	Curvature
Level	1.00000		
Slope	-0.99997	1.00000	
Curvature	-0.99960	0.99953	1.00000
Variance-Covariance	Level	Slope	Curvature
Level	4.3045	-4.2925	-4.6501
Slope	-4.2925	4.2808	4.6370
Curvature	-4.6501	4.6370	5.0276

Table 5 Level, Slope, Curvature Parameter Estimates

(1) US Bond Yield

Optimal τ 1.78561658	Maturity	Level	Slope	Curvature	Pos1 Cash Flow
0.25	CM3m	1	0.933152	0.0638	$1.73
0.5	CM6M	1	0.872195	0.116422	$3.19
1	CM1Y	1	0.765687	0.194496	$17.91
2	CM2Y	1	0.601521	0.275261	$31.85
3	CM3Y	1	0.484285	0.297928	$19.75
5	CM5Y	1	0.33541	0.274609	$9.77
7	CM7Y	1	0.250028	0.230191	$5.79
10	CM10Y	1	0.177902	0.174205	$4.28
20	CM20Y	1	0.08928	0.089266	$4.04
30	CM30Y	1	0.059521	0.05952	$1.68
				Total=	$100.00

(2) Taiwan Bond Yield

Optimal τ 50.0781554	Maturity	Level	Slope	Curvature	Bond Index Cash Flow
0.25	CM3m	1	0.997508	0.002488	$1.92
0.5	CM6M	1	0.995024	0.004959	$3.55
1	CM1Y	1	0.990082	0.009852	$19.90
3	CM3Y	1	0.970636	0.028783	$35.39
5	CM5Y	1	0.951699	0.04672	$21.95
7	CM7Y	1	0.933255	0.063707	$10.86
10	CM10Y	1	0.906483	0.087497	$6.43
				Total=	$100.00

With the three factor parameters and its variance-covariance, using equation (16), we then can estimate the VaR figures of bond yield change for both the US and Taiwan, $2.4642 and $2.4902, respectively as shown in Table 6. The linear level, slope, and curvature three factor model, which uses fewer factors, has the advantage of model parsimoniousness but provides weaker estimates of the bond data nonlinear effects of the convexity and diversification. Therefore, unlike the VaR estimation results of the key rate method that involved more risk factors, the VaR of the US bond data is less than the VaR of the Taiwan bond data.

Table 6 provides the VaR sensitivity measures of the three factors by computing its VaRdelta and VaRbeta. As the table shows, in the US the slope of the bond data has the highest VaR delta and component while in Taiwan the level has the highest VaR delta and component.. This means that the most VaR sensitive is the slope factor for US whereas it is the level factor for Taiwan. This description corresponds to our forgoing economic introduction that in US the slope is weakening, and the level is going down significantly in Taiwan.

Table 6 Level, Slope, Curvature Duration Value at Risk

(1) US Bond Yield

L,S,C Covariance	Level	Slope	Curvature	LSC Cash Flow	VaRDelta	VaR Component	VaRBeta
Level	2.659E-05	4.787E-05	5.451E-05	$100.00	$0.0071	$0.7121	28.90%
Slope	4.787E-05	2.788E-04	1.080E-04	$52.78	$0.0243	$1.2798	51.94%
Curvature	5.451E-05	1.080E-04	2.888E-04	$23.84	$0.0199	$0.4723	19.17%
					Total =	$2.4642	100.00%
					VaR =	$2.4642	

(2) Taiwan Bond Yield

L,S,C Covariance	Level	Slope	Curvature	LSC Cash Flow	VaRDelta	VaR Component	VaRBeta
Level	4.3045	-4.2925	-4.6501	$100.00	$0.5354	$53.5367	2149.90%
Slope	-4. 2925	4.2808	4.6370	$96.35	($0.5085)	($48.9988)	-1967.67%
Curvature	-4. 6501	4. 6370	5.0276	$3.52	($0.5822)	($2.0477)	-82.23%
					Total =	$2.4902	100.00%
					VaR =	$2.4902	

Note: LSC cash flows are obtained by key rate cash flows transformations from the factor coefficients.

3.4 Principal component duration value at risk

Principal component analysis uses the characteristic value of the key rate variance-covariance matrix and correlations and then transforms the structure of variables into several primary components, while maintaining the original total variance or maximizing the transformation total variance. After variable transformation by PCA, the primary components will be independent of each other and normally there will be two or three primary components accounting for most of the variable variance. Compared to the key rate and three-factor yield curve model, the PCA has the advantage of factor independence; factor independence is helpful for investment analysis since we don't need to care about the consequential factor relationship risk. However, the primary components are not easy to describe. Therefore, PCA has been considered as a theoretical tool. On the other hand, unlike the three-factor model that finds the optimal τ by minimizing the forecast error, PCA finds their eigen values and vectors by maximizing the score variance. Thus, PCA wouldn't be used for the time series prediction but used for a system analysis that uses the whole information for the parameter estimates such as the eigen value and vector computation.

By equation (18), we obtain the eigen values and eigen vectors of the variance and covariance of the key rates as shown in Table 7. Table 8 shows that three of the components account for 99.80% of the term structure variance for US bond yields and 98.91% of the term structure variance for Taiwan bond yields.

Further using the PCA cash flow and eigen values only without its variance and covariance structure, we can measure the PCA bond yield VaR figure for both US and Taiwan bond yield since PCA are independent of each other. Then, Table 8 uses the PCA cash flow comprised of the key rate cash flows and the eigen vectors, and eigen volatility to measure the bond yield VaR according to equation (22) for both the US and Taiwan. We obtain the US bond yield data VaR of $2.5104 and the Taiwan bond yield data VaR of $2.506. The higher US bond yield PCA VaR figure means its interest rate system risk is more than Taiwan's. However, the nonlinear effects cannot be discovered by PCA, and thus PCA tends to overestimate the VaR of the Taiwan's bond yield data that exhibits convexity and diversification effects, the same as key rate method mentioned previously. Besides, the PCA treats the time data as cross-sectional data and therefore can't be used to forecast the term structure or the conditional VaR.

Table 7 EigenValues and EigenVectors of the Variance-Covariance Interest Rate Term Structure (page 1 of 2)

(1) US Bond Yield Eigen Values and Eigen Vectors

Eigen Value λ	Eigen Vol	Eigen Vector									
1.9738E-03	4.4427E-02	-0.4256	0.4671	0.3374	-0.3404	-0.328	-0.402	-0.1943	-0.2512	6.47E-03	-0.0296
5.9122E-05	7.6891E-03	-0.4119	0.3715	0.0464	-0.0708	0.0722	0.3527	0.6281	0.3972	-0.0481	0.0317
6.8942E-06	2.6257E-03	-0.4131	0.1995	-0.182	0.097	0.4268	0.3784	-0.6385	0.0131	0.0215	0.116
2.3635E-06	1.5374E-03	-0.3905	-0.0879	-0.3167	0.3464	0.1781	-0.2562	0.2492	-0.3458	0.3126	-0.4927
7.0345E-07	8.3872E-04	-0.348	-0.2165	-0.2811	0.2593	-0.1354	-0.2778	0.107	-0.083	-0.497	0.57
3.7986E-07	6.1633E-04	-0.271	-0.3299	-0.0891	-3.58E-03	-0.48	0.0956	-0.2553	0.4541	-0.2367	-0.4928
1.4716E-07	3.8361E-04	-0.232	-0.3584	-7.92E-03	-0.188	-0.0827	-0.1479	-0.0309	0.3127	0.7121	0.3844
9.8330E-08	3.1358E-04	-0.174	-0.3565	0.0108	-0.3701	-0.2043	0.5556	0.1301	-0.5801	-4.19E-03	0.0418
5.9998E-08	2.4494E-04	-0.1194	-0.3268	0.0887	-0.5497	0.5982	-0.2891	0.0586	0.1054	-0.2971	-0.1568
2.4010E-08	1.5495E-04	-0.1822	-0.2798	0.8097	0.4561	0.1347	0.069	-4.25E-04	-0.03	-0.0319	7.88E-03
Sum of λ=	ai'ai=	1.00004	1.00006	1.00003	1.00005	0.99997	1.00004	0.99999	1.00002	0.99996	1.00000
0.002044	ai aj=	0.00002	0.00007	-0.00003	0.00001	0.00000	-0.00006	-0.00001	-0.00003	0.00001	

(2) Taiwan Bond Yield Eigen Values and Eigen Vectors

Eigen Value λ	Eigen Vol	Eigen Vector						
1.639E-03	0.04049	-0.4049	0.3578	-0.0719	0.044	0.0624	0.3587	0.7539
3.875E-05	0.00622	-0.4065	0.4954	-0.2284	0.0272	-0.5417	-0.4396	-0.2228
2.088E-05	0.00457	-0.3773	0.215	-0.0778	0.0645	0.301	0.5765	-0.615
8.426E-06	0.00290	-0.396	4.86E-03	0.19	0.0465	0.6855	-0.5784	0.0188
6.015E-06	0.00245	-0.3549	-0.2275	0.3751	-0.7996	-0.1913	0.0726	-0.0188
3.452E-06	0.00186	-0.3447	-0.6214	-0.701	2.44E-03	-0.0195	-0.0206	0.0542
8.134E-07	0.00090	-0.3562	-0.3774	0.5181	0.593	-0.3242	0.0742	-5.83E-03
Sum of λ=	ai'ai=	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9999
0.001717	ai'aj=	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	

Table 8 PCA Cash Flows and VaR

(1) US Bond Yield

Maturity	Spot Rate	VaR(%)	Bond Index	EigenValueλ	PC Vol	% Vol Explained	% Vol Cummulated	PCA Cash Flow (v_i)	$v_i^2 * \lambda$
CM3m	2.37%	3.172%	$1.73	1.9738E-03	4.443%	96.58%	96.58%	-34.286	2.32031
CM6m	2.61%	3.048%	$3.19	5.9122E-05	0.769%	2.89%	99.48%	-10.124	0.00606
CM1Y	2.87%	3.033%	$17.91	6.8942E-06	0.263%	0.34%	**99.82%**	-17.318	0.00207
CM2Y	3.24%	2.862%	$31.85	2.3635E-06	0.154%	0.12%	99.93%	12.913	3.941E-04
CM3Y	3.41%	2.563%	$19.75	7.0345E-07	0.084%	0.03%	99.97%	6.901	3.350E-05
CM5Y	3.71%	2.026%	$9.77	3.7986E-07	0.062%	0.02%	99.98%	-5.034	9.627E-06
CM7Y	3.97%	1.757%	$5.79	1.4716E-07	0.038%	0.01%	99.99%	-1.594	3.738E-07
CM10Y	4.23%	1.355%	$4.28	9.8330E-08	0.031%	0.00%	100.00%	-7.447	5.454E-06
CM20Y	4.76%	0.980%	$4.04	5.9998E-08	0.024%	0.00%	100.00%	0.915	5.024E-08
CM30Y	4.64%	1.426%	$1.68	2.4010E-08	0.015%	0.00%	100.00%	-5.340	6.846E-07
Total=			$100.00	0.002044		100.00%		-60.414	2.3289
								VaR=	$2.5104

(2) Taiwan Bond Yield

Maturity	Spot Rate	VaR(%)	Bond Index	EigenValue λ	PC Vol	% Vol Explained	% Vol Cummulated	PCA Cash Flow (v_i)	$v_i^2 * \lambda$
CM3m	0.9962%	2.727%	$1.92	1.639E-03	4.049%	95.44%	95.44%	-37.567	2.31325
CM6M	0.9722%	2.773%	$3.55	3.875E-05	0.622%	2.26%	97.70%	-7.272	0.00205
CM1Y	1.6975%	2.535%	$19.90	2.088E-05	0.457%	1.22%	**98.91%**	8.178	0.00140
CM2Y	3.1279%	2.663%	$35.39	8.426E-06	0.290%	0.49%	99.40%	-10.602	0.00095
CM3Y	3.3233%	2.424%	$21.95	6.015E-06	0.245%	0.35%	99.75%	21.952	0.00290
CM5Y	4.4866%	2.440%	$10.86	3.452E-06	0.186%	0.20%	99.95%	-8.022	0.00022
CM7Y	3.0960%	2.456%	$6.43	8.134E-07	0.090%	0.05%	100.00%	-10.773	0.00009
Total=			$100.00	0.001717		100%		-44.107	2.32086
								VaR=	$2.506

Note: PCA cash flows are obtained by key rate cash flows transformation from eigen vectors.

3.5 Structural Equation Yield Curve Model

Principal component analysis explores the data reducing process to find the primary component variances that maximize the total variances after variable transformation. Unlike PCA that explores the primary components with entire factor loading estimates, structural equation modeling (SEM) tries to control the specifications of the constructs with partial factor loading estimates that are the measurement model of the SEM, i.e. the construct or factor measurements. Thus, in our SEM analysis of the interest rate term structure, we will subjectively try to find the controlled constructs that can be measured by our indicators- the structural key rates with different maturities. We will use LISREL as our SEM solver. The four steps involved are as follows:

Step One: Find Construct Measurement

Considering our 10 key rates for the US bond data ranging from 3 months to 30 years and the modification indices obtained from LISREL, we suggest splitting the term structure into short-term, medium-term, and long-term constructs as Table 9-(a) shows for the SEM confirmatory factor analysis. Construct one has 3-month, 6-month, and 1-year maturities; construct 2 has 2-year, 3-year, 5-year and 7-year maturities; construct 3 has 10-year, 20-year, and 30-year maturities. As for Taiwan bond yield, after examining the modification indices, chi-square, and root mean square residuals by adding and deleting factors, we suggest using three constructs consisting of the key rates ranging from 3 months to 10 years: hence construct one has 3-month and 6-month maturities, construct two has 1-year and 2-year maturities, and finally construct three has 3-year, 5-year, and 7-year maturities as shown in Table 9-(b).

Table 9 Confirmatory Factor Analysis SEM measure model Notations

(a) US Constructs

Exogenous Indicator (Key Rates)	Exogenous Constructs	Error
X1(CF3m)	$\lambda_{11}\ \xi_1$	δ_1
X2(CF6m)	$\lambda_{21}\xi_1$	δ_2
X3(CF1y)	$\lambda_{31}\xi_1$	δ_3
X4(CF2y)	$\lambda_{12}\ \xi_2$	δ_4
X5(CF3y)	$\lambda_{22}\xi_2$	δ_5
X6(CF5y)	$\lambda_{32}\xi_2$	$\delta 6$
X7(CF7y)	$\lambda_{42}\xi_2$	δ_7
X8(CF10y)	$\lambda_{13}\xi_3$	δ_8
X9(CF20y)	$\lambda_{23}\xi_3$	δ_9
X10(CF30y)	$\lambda_{33}\xi_3$	δ_{10}

(b) Taiwan Constructs

Exogenous Indicator (Key Rates)	Exogenous Constructs	Error
X1(CF3m)	$\lambda_{11}\ \xi_1$	δ_1
X2(CF6m)	$\lambda_{21}\xi_1$	δ_2
X3(CF1y)	$\lambda_{31}\xi_1$	δ_3
X4(CF2y)	$\lambda_{12}\ \xi_2$	δ_4
X5(CF3y)	$\lambda_{22}\xi_2$	δ_5
X6(CF5y)	$\lambda_{13}\xi_3$	δ_6
X7(CF7y)	$\lambda_{23}\xi_3$	δ_7

Step Two: Data Input

SEM tries to search the appropriate relationships (loadings) of the factors and indicators. Therefore, the data input for the SEM model estimation is the variance-covariance decomposition of the covariance matrix or the correlation matrix. Though the correlation has been used as the measure unit, we should use the covariance as the input data for the variance research needed for our value at risk analysis.

Step Three: Optimal Model Estimation

Like multivariate data analysis methods such as linear multivariate regression, multivariate logit or probit regression, and even the exploratory factor analysis and principal component analysis, the parameters of the SEM can be estimated by ordinary least squares (OLS) or maximum likelihood estimation (MLE).

Step Four: Testing Model Estimates

Since MLE is one of the estimation methods, the likelihood ratio chi-square (χ^2), Wald statistics, and Lagrange multiplier can be used to measure the model fit. Other approaches provided by LISREL are the goodness of fit index and the root mean square error. Table 10 shows the confirmatory construct factor loadings for each index (observable) measure. The t statistics of the loadings and the R^2 of the construct measures exhibit high significance of the coefficients and good variance explanation of the measure model. In Table 10, the chi-square values are very large numbers- 12233.81 for US, and 675.61 for Taiwan, so the model does not fit the covariance of the bond data very well. Nevertheless, our aim is to find the appropriate factor loadings and significances for the variance and covariance decomposition.

Finally to compute the value at risk for the interest rate term structure of the SEM, we use LISREL to estimate the correlations of the three factors as shown in Table 11.

In addition, the variances of the three factors also can be obtained by bond data transformation into factor data and computing factor data variances. With the correlation of the factors and their estimated variances, we can calibrate the SEM VaR figure as done in the PCA model and the three-factor model- level, slope, and curvature. We describe the mathematics of the SEM VaR calibration as follows:

Table 10 Confirmatory Factor Loading Tests

(a) US Bond Yield t Statistics

CM3m = 0.01923*Factor1, Errorvar.= 0.0000 , R^2= 0.9941	
(0.0003530)	(0.0000)
54.4676	27.6341
CM6M = 0.01855*Factor1, Errorvar.= -0.0000 , R^2= 1.0025	
(0.0003378)	(0.0000)
54.9262	-21.2627
CM1Y = 0.01834*Factor1, Errorvar.= 0.0000 , R^2= 0.9899	
(0.0003382)	(0.0000)
54.2414	29.0438
CM2Y = 0.01735*Factor2, Errorvar.= 0.0000 , R^2= 0.9947	
(0.0003184)	(0.0000)
54.5017	31.4205
CM3Y = 0.01560*Factor2, Errorvar.= -0.0000 , R^2= 1.0020	
(0.0002841)	(0.0000)
54.9011	-25.8755
CM5Y = 0.01223*Factor2, Errorvar.= 0.0000 , R^2= 0.9860	
(0.0002264)	(0.0000)
54.0315	31.9571
CM7Y = 0.01051*Factor2, Errorvar.= 0.0000 , R^2= 0.9680	
(0.0001980)	(0.0000)
53.0698	30.7898
CM10Y = 0.008262*Factor3, Errorvar.= -0.0000 , R^2= 1.0068	
(0.0001498)	(0.0000)
55.1568	-10.3380
CM20Y = 0.005843*Factor3, Errorvar.= 0.0000 , R^2= 0.9612	
(0.0001109)	(0.0000)
52.7029	26.3641
CM30Y = 0.008225*Factor3, Errorvar.= 0.0000 , R^2= 0.8996	
(0.0001660)	(0.0000)
49.5568	28.3992

Note:
1. Chi-Square=12233.81(df=32), P value=0.00000, and RMSE=0.504.
2. Numbers in parenthesis are standard deviations.

(b) Taiwan Bond Yield t Statistics

CM3m = 0.01661*Factor1, Errorvar.= -0.0000 , R^2= 1.0043	
(0.0003020)	(0.0000)
55.0222	-10.0113
CM6M = 0.01663*Factor1, Errorvar.= 0.0000 , R^2= 0.9732	
(0.0003118)	(0.0000)
53.3406	25.4817
CM1Y = 0.01535*Factor2, Errorvar.= 0.0000 , R^2= 0.9925	
(0.0002823)	(0.0000)
54.3791	14.8548
CM3y = 0.01595*Factor2, Errorvar.= 0.0000 , R^2= 0.9715	
(0.0002996)	(0.0000)
53.2455	25.4609
CM5y = 0.01447*Factor3, Errorvar.= 0.0000 , R^2= 0.9641	
(0.0002740)	(0.0000)
52.8002	16.9407
CM7Y = 0.01394*Factor3, Errorvar.= 0.0000 , R^2= 0.8835	
(0.0002866)	(0.0000)
48.6578	24.5697
CM10Y = 0.01456*Factor3, Errorvar.= 0.0000 , R^2= 0.9509	
(0.0002794)	(0.0000)
52.1089	19.8596

Note:
1. Chi-Square=675.61 (df=11), P value=0.00000, and RMSE=0.201.
2. Numbers in parenthesis are standard deviations.

Table 11 SEM Confirmatory Factor Correlation

(a) US Factor Correlation (Covariance)

	Factor1	Factor2	Factor3
Factor1	1		
Factor2	0.9507	1	
	(0.0024)		
	403.6696		
Factor3	0.8687	0.9606	1
	(0.0062)	(0.0019)	
	141.1978	506.1870	

Note: In each factor block, standard deviation is in parenthesis and t statistics is at the bottom.

(b) Taiwan Factor Correlation (Covariance)

	Factor1	Factor2	Factor3
Factor1	1		
Factor2	0.9951	1	
	(0.0004)		
	2348.508		
Factor3	0.9599	0.9779	1
	(0.0022)	(0.0015)	
	427.9668	644.9827	

$$\Delta P/P = \Delta P/P \mid f1 + ... + \Delta P/P \mid f2, \qquad (25)$$

where $\Delta P/P$ is the total percentage price change (bond yield) and $\Delta P/P \mid fi$ is the change in price due to factor *i* (the SEM factors),

$$\sigma^2(\Delta P/P) = \text{factor cash flow} * \Omega * \text{factor cash flow}' \qquad (26)$$

where Ω is the factor variance-covariance matrix and factor cash flow is the investment position allocated to the specific factor. Table 12 shows the factor cash flows for three factors (short, medium, and long term) according to our factor loadings. Table 13 presents the VaR estimates with the confirmatory factor SEM method.

Table 12 SEM Confirmatory Factor Cash flows

(a) US Bond Yield Factor Cash Flows

Maturity	Factor Coefficients Factor1	Factor2	Factor3	Key Rate Cash Flow
CM3m	0.0192	0	0	$1.73
CM6M	0.0186	0	0	$3.19
CM1Y	0.0183	0	0	$17.91
CM2Y	0	0.0174	0	$31.85
CM3Y	0	0.0156	0	$19.75
CM5Y	0	0.0122	0	$9.77
CM7Y	0	0.0105	0	$5.79
CM10Y	0	0	0.0083	$4.28
CM20Y	0	0	0.0058	$4.04
CM30Y	0	0	0.0082	$1.68
			Total=	$100.00
	Factor1	Factor2	Factor3	
Factor Cash Flow	0.4210	1.0411	0.0728	

Note: Factor cash flows are obtained using key rate cash flows transformation by the factor coefficients.

(b) Taiwan Bond Yield Factor Cash Flows

Maturity	Factor Coefficients Factor1	Factor2	Factor3	Key Rate Cash low
CM3m	0.01661	0	0	$1.92
CM6M	0.01663	0	0	$3.55
CM1Y	0	0.01535	0	$19.90
CM2Y	0	0.01595	0	$35.39
CM3Y	0	0	0.01447	$21.95
CM5Y	0	0	0.01394	$10.86
CM7Y	0	0	0.01456	$6.43
			Total=	$100.00
	Factor1	Factor2	Factor3	
Factor Cash Flow	0.0910	0.8699	0.5626	

Table 13 Value at Risk with SEM Confirmatory Factor Estimate

(a) US Bond Yield VaR

Factor Covariance (Correlation)	Factor1	Factor2	Fctor3	CFA Cash Flow	VaRDelta	VaR Component	VaRBeta
Factor1	1.0000	0.9507	0.8687	$0.421	$1.5993	$0.6733	27.00%
Factor2	0.9507	1.0000	0.9606	$1.041	$1.6397	$1.7071	68.45%
Factor3	0.8687	0.9606	1.0000	$0.073	$1.5609	$0.1137	4.56%
			Total=	$1.535		$2.4941	100.00%
			VaR=	$2.4941			

(b) Taiwan Bond Yield VaR

Factor Covariance (Correlation)	Factor1	Factor2	Factor3	CFA Cash Flow	VaRDelta	VaR Component	VaRBeta
Factor1	1	0.9951	0.9599	$0.091	$1.6253	$0.1479	5.93%
Factor2	0.9951	1	0.9779	$0.870	$1.6405	$1.4271	57.27%
Factor3	0.9599	0.9779	1	$0.563	$1.6296	$0.9168	36.79%
			Total=	$1.523		$2.4918	100.00%
			VaR=	$2.4918			

We should note that the three factors of the SEM have entirely different descriptions from the three factors of the level, slope, and curvature model. The level, slope, and curvature model uses all of the key rates as their factor constructs, but the SEM uses only part of the key rates as their factor constructs as seen in Table 5 and Table 9. As for the model fit, the data model fit of the SEM performs ineffectively and also it is not a good time series predictor, whereas the level, slope, and curvature factor model finds its optimal fit by the root mean square forecast errors and thus it is a better tool for yield prediction.

For the Taiwan bond data, the three-factor SEM VaR measure is approximately the same ($2.4918) as the level, slope, and curvature VaR measure ($2.4902) in Table 6; both of the factor models- SEM and level, slope, and curvature- models measure part of the total variance ignoring the residual variances. However, for the US bond data VaR, the three-factor SEM VaR measure has a slightly different result than from the level, slope, and curvature due to the constraints of the constructs set by the SEM. In addition, using SEM, we see higher VaR sensitivity of the medium-term yield for both the US and Taiwan bond data. Using the level, slope, and curvature model, we observe higher VaR sensitivity of the slope for US and the level for Taiwan. This can be explained by the higher factor correlations between medium and both the short and long-term yields and the higher cash position allocated in the medium-term yield.

On the other hand, while both the SEM and three-factor model have some degree of correlation between their factors, the PCA components are independent of each other and have a divergent VaR figure ($2.5104 for US and $2.506 for Taiwan) from the factor models-SEM ($2.4941 for US and $2.4918 for Taiwan) and three-factor model ($2.464 for US and $2.490 for Taiwan). This divergence can be explained by the residuals of the model fit of the factor models, such as

SEM and the level, slope, and curvature model. When we use data analysis models that do not use any estimation procedure to fit the model like the factor models, we find that the PCA VaRs ($2.5104 for US and $2.506 for Taiwan) have similar estimates as the key rate method VaRs ($2.511 for US and $2.507 for Taiwan).

4. Conclusion

Key rate duration application is the primary methodology used by JP Morgan for its bond portfolio value at risk measure. In reality, although there exists the issues of capturing the real world key rate variance-covariance structure and yield curve movement, many interest rate risk managers still prefer to use it for its simple and detailed descriptions of real factor definitions. However, for interest rate risk researchers, the unstable key rate variance-covariance and correlation structure and the unknown yield curve current movement often limit its use for long term interest rate risk management.

The three factor--level, slope and curvature--interest rate model, might not fit the general needs of interest rate risk management. It tends to underestimate the VaR figure. Nevertheless, because of the distinctive characteristics of the three factors and its parsimonious features, this yield curve model should have some advantages as well. Especially, when considering the economic changes of inflation, business cycle, and economic volatility, the three factors--level, slope, and curvature—model would perform well at describing the yield curve movement. For instance, we can explain the three-factor model effect upon the bond portfolio risks such as the barbell and bullet bond portfolios. When there is a higher term structure level or curvature change, the higher cost barbell bond would have a lower bond value at risk since it has a larger maturity difference (i.e. convexity) and a lower correlation between maturities. When there is a higher term structure slope change, the bullet bond would have a lower bond value at risk. Empirically, the US bond data exhibits a higher slope risk whereas Taiwan bond data reveal a higher level risk according to the level, slope, and curvature model.

If we want to analyze only two or three specific component interest rates and forgo the component interest rate variance-covariance and correlation structures, PCA is a good method of VaR estimation owing to its factor independences and variable reduction. However, the explanation of the components is not easy for researchers, not to mention investors. Some researchers tend to explain its components as the level, slope, and curvature but the PCA factors should be independent and the level, slope, and curvature would likely have some kind of correlation among them.

Concerning SEM, it is primarily used for the confirmatory factor analysis. In addition to the exploratory factor analysis that searches for the optimal factor loading by achieving the maximum likelihood or least square objective, SEM needs the actual perception of the researchers in the field of factor constructs (measurements) to find the appropriate measurement and structural models. Although in this research, SEM does not fit the data model well (high chi-square and small p value), we just aimed to find the factor loadings (variance decomposition) and use the covariance and correlation as the key input data for SEM analysis. Compared to three-factor model, SEM has done equally well for the VaR measure despite the model's disappointing fit to the data. Nonetheless, in practice, researchers want to construct good and easily interpreted factors for investor risk management as in our case: the short, medium, and long term VaR measures that can be recognized painlessly, and VaR components as well as sensitivity analysis. Empirically, the US bond data and Taiwan bond data have higher medium term VaR sensitivities, and US bond data VaR is slightly more than Taiwan bond data VaR according to SEM.

In sum, principal component analysis and factor models such as the level, slope, and curvature model and SEM are parsimonious models that use fewer variables to explain the data change behavior while key rate uses more risk variables. In particular, the risk factors of the SEM and level, slope, and curvature model are easily understood and useful for the interest rate risk analysis. However, both of the factor models tend to underestimate the VaR figures slightly because of doing the model fit and bypassing the residual variances. On the other hand, the data analysis models such as the key rate and PCA methods perform the variance/covariance decomposition and combination to find out the bond yield risk and risk components.

References

[1] Austin and R.F. Calderon. (1996), "Theoretical and Technical Contributions to Structural Equation Modeling*: An Updated Annotated Bibliography," Structural* Equation *Modeling* 3:105-175.

[2] Bodurtha, Jr., James N. and Qi Shen. (1994), "Historical and Implied Measures of Value at Risk: The DM and Yen Case," Working Paper, McDonough School of Business, Georgetown University.

[3] Fisher, L. (1966), "An Algorithm for Finding Exact Rates of Return", *The Journal of Business* 39/1, January, pp. 111-118.

[4] Golub, W. and Tilman, M. (1997), "Measuring Yield Curve Risk Using Principal Components Analysis, Value at Risk, and Key Rate Durations," *The Journal of Portfolio Management*.

[5] Ho, T.S.Y. (1992), "Key Rate Durations: Measures of Interest Rate Risks," *The Journal of Fixed Income*. New York: Vol. 2, Iss. 2; p. 29 (16 pages).

[6] JP Morgan/Reuters. (1996), "RiskMetrics™ - Technical Document."See http://www.riskmetrics.com/techdoc.html#rmtd

[7] Khindanova, Irina N. and Svetlozar T. Rachev (2002), "VALUE AT RISK: RECENT ADVANCES," Working Paper, University of California, Santa Barbara and University of Karlsruhe, Germany.

[8] Nelson, R. and Siegel, F. (1987), "Parsimonious Modeling

of the Yield Curve," *Journal of Business*, Vol. 60, No.

[9] Singh, K. (1997), "Value at Risk Using Principal Components Analysis," *The Journal of Portfolio Management*, New York: Vol. 24, Iss. 1; p. 101 (12 pages).

[10] Tremblay, P.F. & Gardner, R.C. (1995), "Expanding the motivation construct in language learning," *The Modern Language Journal*, 79/4, 505-518.

[11] Venkatesh, P.C. (2003), "Value at risk for corporate bond portfolios," *The Journal of Fixed Income*, New York: Vol.13, Iss. 2; pg. 19.

[12] Vlaar, Peter J. G. (2000), "Value at risk models for Dutch bond portfolios," *Journal of Banking & Finance* Volume 24, Issue 7, Pages 1131-1154.

[13] Willner, R. (1996), "A New Tool for Portfolio Managers: Level, Slope, and Curvature Durations," *Journal of Fixed Income* New York: Jun 1996. Vol. 6, Iss. 1; p. 48 (12 pages).

[14] Wilson, D. (1995) "VaR in Operation," *Risk,* 1995, 8, 24-25.