Determining Reorder Point in the Presence of Stochastic Lead Time and Box-Jenkins Time Series Demand

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Abstract

This paper discusses the determination of reorder point and safety stock when the period demands are not independent, but exhibits a serially correlated demand process which can be represented as an autoregressive-moving average – ARMA (p.d.q)-model for both deterministic and discrete stochastic lead time. An Excel based methodology for finding the reorder point and the safety stock level is also presented at the end of the paper.

1. Introduction

Generally when the determination of the optimal policy of an inventory model with a stochastic demand includes the calculation of the reorder point and the order size, one has to deal with mean rate of demand, standard deviation, safety factor, and lead-time. Mostly, the calculation of the re-order point is based on the assumption that the mean rate demand is deterministic as a function of time. The deterministic assumption on the mean rate of demand is by far remote from the reality. Therefore, it will be more appropriate to use appropriate probability distributions to represent the lead-time and the units demanded to account for the increasing uncertainty in the market environment. Based on the assumption that there is no correlation between two period demands, Hadley and Whitten [6] have developed two types of backorder inventory models, approximate and exact ones, for both Poisson and Normal distributed lead-time demands. In many practical situations, however, the period demands are not independent, but exhibit a serially correlated process (see, e.g. An, Fotopolo, and Wang [1]; Charles, Marmorsten, and Zinn [3]).

Taking into consideration (1) the rate of demand, (2) the length of lead-time, (3) the variability of demand and lead-time, and (4) the degree of acceptable stock-out risk, Eppen and Martin [5] suggested that the correct and consistent procedure to set reorder points with assumed data is to: (1) clearly distinguish between demand variation and the variation in forecast error, (2) show how to calculate the variance of forecast error over lead time without assuming forecast errors are normally distributed, (3) use the variance of forecast error over lead time to set the safety stock, and (4) show the calculation of safety stock can be simplified if normality of cumulative forecast error is justified as in the case when the process generating demand is from the Box and Jenkins time series data.

In this research paper, we look into the analysis that incorporates the calculations of reorder point and safety stock when the units demanded are generated by a serially correlated process and can be represented by Box-Jenkins' ARMA time series model [2]. The distribution of forecast errors from the calculation process in Box-Jenkins' ARMA analysis will be used as the measurement of the accuracy with which the reorder point and safety stock are determined. In the first part of this research paper, the determination of the model's reorder point is based on the assumption that the procurement lead-time is a random variable generated by an ARMA process with constant lead-time. In the second part, we would investigate the case of stochastic lead time.

2. Two Different Types of Lead-Time Demand Distributions

In order to compute the reorder point with safety stock that will meet a specific service level, one has to know the probability density of the lead time demand. Two kinds of joint probability density functions can be used to represent the lead time demand.

2.1. $p_{1,\ldots,j}(z_1,\ldots,z_j)$

If we use

$$L_j = Z_{t+1} + Z_{t+2} + \ldots . Z_{t+j} \quad (1)$$

as a representation of the demand during the lead time of *J* periods, then

$$E[L_t] = j\mu \quad (2)$$

$$Var[L_t] = \sigma_a{}^2 \sum_{s=1}^{n} \sum_{t} \gamma_{|t-s|} \quad (3)$$

where $\gamma_k = Cov[Z_t - Z_{t-k}]$ is the autocovariance at lag *K*.

2.2. $p_{1,\ldots,j}(z_{t+1},\ldots,z_{t+j} | z_t, z_{t-1},\ldots,z_{t-n})$

$p_{1,\ldots,j}(z_{t+1},\ldots,z_{t+j} | z_t, z_{t-1},\ldots,z_{t-n})$ is the conditional probability distribution for the period demand during the lead time. If we define the lead time demand as the future value as

$$L_t(j) = Z_{t+1} + Z_{t+2} + \ldots\ldots . Z_{t+j} \quad (4)$$

given that we have observed the past values $Z_t, Z_{t-1},\ldots, Z_{t-n}$ which occurred prior to the period time *t*, then we will have the expected value and variance involved in the reorder point calculation as

$$E[L_t(j) | Z_t, Z_{t-1},\ldots, Z_{t-n}] = \hat{Z}_t(1) + \hat{Z}_t(2) + \ldots \hat{Z}_{t-j}(j) \quad (5)$$

where $\hat{Z}_t(j)$ is the forecast value, and

$$Var[L_t(j)] = \sigma_a{}^2 \{\sum_{i=1}^{j} g_{i,i} + 2\sum_{i=1}^{j} \sum_{k>i}^{j} g_{i,k}\} \quad (6)$$

where $Cov[L_t(j), L_t(j+k)] = \sigma_a^2 \sum_{i=0}^{j-1} \psi_i \psi_{k+i} = \sigma_a^2 g_{j,j+k}$, and (7)

$g_{j,j+k}$ is the autocorrelation of the forecast errors between the period *J*, and *J* + *K*.

If the demand is believed to be essentially represented by the Box and Jenkins time series process, then the demand forecast error for period *t* + *j* based on data through period t is

$$e_t(j) = Z_{t+j} - \hat{Z}_t(j) \quad (8)$$

where $\hat{Z}_t(j)$ is the minimum mean squared error forecast selected from one of the Box and Jenkins' models for j periods from the origin t.

We can see the difference between the first and second moments of the two kinds of distributions. Eppen and Martin [5] have shown that using the first and second moments from the first kind of probability distribution led to the inconsistent result of the reorder point value. As pointed out by Chopra, Gilles and Maqbool [4], managers have been under increasing pressure to reduce inventories to streamline the supply chain. Their goal is to reduce inventories without hurting the level of service provided to customers. Safety stock, however, is a function of the cycle service level, the demand uncertainty, the replenishment lead time, and the lead time uncertainty. In this paper, we demonstrate how to compute the reorder point and the safety stock level when the lead time demand can be appropriately represented by a Box-Jenkins Time Series Model.

3. Deterministic Lead Time Of J Periods

The analysis in this section proceeds as suggested by Eppen and Martin [5], i.e., first specify the service level (α) given a *j*-period lead time. Then choose a reorder point *R* such that the probability of stocking out during the lead time does not exceed α. That is, select the smallest *R* such that

$$\Pr[\sum_{i=1}^{j} Z_{t+i} > R] \le \alpha \quad (9)$$

The first step in selecting *R* is to forecast the demand during the lead time. That means that at the time *t*, forecasts are required for periods *t* = *j* for *j* = 1, 2, . . ., then use the forecast error probability during the lead time to select the reorder point, i.e.,

$$\Pr[\sum_{i=1}^{j} Z_{t+i} - \sum_{i}^{j} \hat{Z}_t(i) > R - \sum_{i}^{j} \hat{Z}_t(i)] \le \alpha \tag{10}$$

$$\Pr[\sum_{i=1}^{j} e_t(j) > a] \le \alpha \tag{11}$$

where $R - \sum_{i}^{j} \hat{Z}_t(i) = a$, the safety stock.

If we let $U_t(j) = \sum_{i=1}^{j} e_t(j)$, the total forecast error during the *j*-period lead time immediately following period *t*, then according to the definition of forecast error defined in the Box and Jenkins' time series process, it follows that $U_t(j)$ is normally distributed with

$$E[U_t(j)] = 0 \tag{12}$$

and

$$Var[U_t(j)] = \sum_{i=1}^{j} Var[e_t(j)] + 2\sum_{k=2}^{j}\sum_{i=1}^{k-1} Cov[e_t(i), e_t(k)] \tag{13}$$

where

$$Cov[e_t(h), e_t(h+b)] = \sigma_a^2 \sum_{i=0}^{h-1} \psi_i \psi_{b+i} = \sigma_a^2 g_t(h, h+b) \tag{14}$$

is the covariance between the *t*-origin forecasts at lead times *h* and *h* + *b*, the value of σ_a is obtained from an estimate of the process residual standard deviation using time series data. $\psi_1, \psi_2, \ldots$ are called the error learning coefficients calculated directly by equating coefficients of B from the following equation.

$$(1 - \varphi_1 B - \varphi_2 B^2 - \ldots)(1 + \psi_1 B^1 + \psi_2 B^2 + \ldots) = (1 - \theta_1 B - \theta_2 B^2 - \ldots) \tag{15}$$

where φ_j is the parameter of the autoregressive term Z_{t-j}, and θ_j is the parameter of the moving average term a_{t-j}.

Using the definition of the normal inverse function, the reorder point level for a given α service level is

$$\frac{R - \sum_{i=1}^{j} \hat{Z}_t(i)}{\Omega_t(j)} = NORMINV(\alpha, 0, \Omega_t(j)) \tag{16}$$

where $\Omega_t(j) = \sqrt{Var(U_t(j))}$ (17)

Another approach to determine reorder point is requiring the percentage of orders filled on time be greater than or equal to β which is defined as the fill rate of the system

$$\beta = 1 - EB_t(j)/Q \tag{18}$$

where *Q* is the order quantity, and $EB_t(j)$ is the expected shortage per cycle. Its formula is

$$EB_t(j) = \Omega_t(j)[Normdist(k_j, 0,1,0) - k_j(1 - k_j Normdist(k_j, 0,1,1))] \tag{19}$$

where $k_j = \dfrac{R - \sum_{i=1}^{j} \hat{Z}_t(i)}{\Omega_j}$ (20)

4. Stochastic Lead Time

If we assume that the lead time random variable takes on the values j $(j = 1, 2, 3, \ldots)$ with probability P_j. If f_j (.) is the density function for U_j then the density of demand during the lead time, is given by

$$f(U) = \sum_{j=1}^{n} f_j(U_j) p_j \tag{21}$$

and

$$F(U) = \sum_{j=1}^{n} F_j(U_j) p_j \tag{22}$$

To insure that the probability of no more than α percent of stocking out during the lead time, R is selected to be the smallest number such that

$$1 - F(R) = 1 - \sum_{j=1}^{n} F_j(R) p_j \le \alpha \tag{23}$$

If we define the standard deviation of the $U_t(j)$ to be $\Omega_t(j)$, then the Excel formula for the service level is

$$\sum_{j=1}^{n} NORMDIST(a, 0, \Omega_t(j)) p_j \le 1 - \alpha \tag{24}$$

The formula for the fill rate with a stochastic lead time is

$$\beta = 1 - \frac{\sum_{j=1}^{n} EB_t(j) p_j}{Q} \tag{25}$$

where $EB_t(j)$ is defined in (19).

For a backorder model, the expected total annual inventory system cost consist of the sum of ordering cost and holding cost (see Hadley and Whittin [6], and Silver and Peterson [7]). If the annual demand is W, and the cost per order is A, then the annual ordering costs are $(W/Q)A$. If the annual holding costs per unit is IC, and the expected mean lead time is $d = \sum_{j=1}^{n}\sum_{i=1}^{j} Z_t(i)$, the holding cost can be represented as $(Q/2 + r - d)IC,$ where $Q/2$ is the average inventory, and $r\text{-}d$ is the average level of safety stock. If β is the fill rate, then the planner is interested in meeting a predetermined fraction of units (β) off the shelf. Then the planner can use the solution of the following model to find the optimal policy.

$$\text{Minimize} \quad TC(Q, r) = Min(W/Q)A + (Q/2 + r - d)IC \tag{26}$$

Subject to

$$\frac{\sum_{j=1}^{n} EB_t(j) p_j}{Q} \le 1 - \beta \tag{27}$$

5. Illustration Of The Computations Of The Forecast $\hat{Z}_t(j)$ And $Var[U_t(j)]$

This section illustrated the computations of $\hat{Z}_t(j)$ and $Var[U_t(j)]$. To obtain the forecast $\hat{Z}_t(j)$, one writes the model in difference form, and use the following rules:

1. Use the available data to compute the known random shocks $a's$, the one-step ahead forecast errors from $a_t = Z_t - \hat{Z}_t(1)$. Note that a_t which is related to the unavailable Z_t data will be assigned a value of zero.
2. Leave $Z_{t-j}(j=1,2,...)$ unchanged because they already happened at origin t.
3. Replace $\hat{Z}_{t+j}(j=1,2,...)$ with their forecasts $\hat{Z}_t(j)$ at origin t because they have not happened.
4. Let $a_{t+j}(j=1,2,...)$ be zero because they have not yet happened.

For expository purpose, the following time series model and data are used in Figure 1 to illustrate the computations using our designed Excel Templates. Suppose that the lead-time demand can be represented by an ARIMA(2,2) model as $Z_t - 1.6Z_{t-1} + 0.62Z_{t-2} = a_t - 0.82a_{t-1} + 0.42a_{t-2}$, with the data for $Z_{47}, Z_{48}, Z_{49}, Z_{50}$, and $\sigma_a = 5.78$.

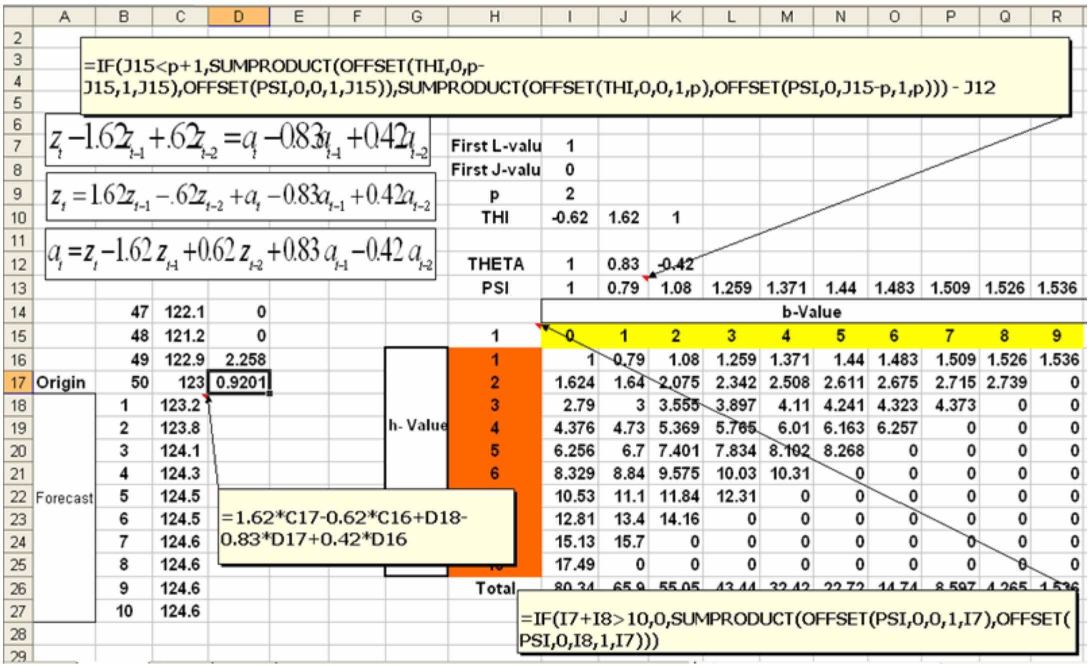

Figure 1 Excel Template for Computing $Z_t(i)'s$ Values and ψ_l's Weights

5.1. Computation of the Forecast values $Z_t(j)'s$

(1) Enter the following available data $Z_{47} = 122.1, Z_{48} = 121.2, Z_{49} = 122.9, Z_{50} = 123$ in cells C14:C17.

(2) Use the equation $a_t = Z_t - 1.62Z_{t-1} + 0.62Z_{t-2} + 0.83a_{t-1} - 0.42a_{t-2}$

to compute a_{49}. Since there were no data available to compute a_{47} and a_{48}, we equated them to zeroes, their expected values. The cell formula for a_{49} in cell D16 is = C16 – 1.62*C15 + 0.62*C14 + 0.83*D15 – 0.42*D14 = 2.258. It was extended to cell D17 to find a_{50}. Since the forecast origin is at t = 50, a_t for t = 51, 52, 53, … in cells D18, …, D27 are the future values of a_t's which have not occurred and thus are given the values of zeroes.

(3) Generate the forecast values $Z_{50}(1), Z_{50}(2), ..., Z_{50}(10)$ using

$Z_t = 1.62Z_{t-1} + 0.62Z_{t-2} + a_t - 0.83a_{t-1} + 0.42a_{t-2}$. The formula in C18 to compute $Z_{50}(1)$ is = 1.62*C17 – 0.62*C16 + D17 – 0.83*D16 + 0.42*D15. The values for $Z_{50}(2), ..., Z_{50}(10)$ can be obtained by extending the formula.

5.2. Computation of $Var[U_t(j)]$

We have developed the Excel formulae to compute the learning error weights for the ARIMA model.

(1) Before using our Excel formulae, the time series model has to be expressed in the form of Equation (15), then use
= SUMPRODUCT(OFFSET(THI,0,p-j,1,j),OFFSET(PSI,0,0,1,j))
to compute ψ_j for j = 1, 2, …, p. and use
= SUMPRODUCT(OFFSET(THI,0,0,1,p +1),OFFSET(PSI,0,j - p,1,p)
to compuite ψ_j for j = p+1, ……………N, where THI, THETA, and PSI are defined as the names for the row vectors ($\varphi_p, \varphi_{p-1}, \ldots, \varphi_1, 1$), ($1, \theta_1, \ldots, \theta_q$), and ($1, \psi_1, \ldots, \psi_s$), respectively.

(2) Use the ψ_j - weight from (2.1.) to compute $g_t(h, h+b) = \sum_{i=0}^{h-1} \psi_i \psi_{b+i}$ of Equation (14). We employed Data/Table to generate the values of $g_t(h, h+b)$ using ***b*** = 1, 2, .., ***h***-1 for the row, ***h*** = 1, 2, …, L for the column, and = SUMPRODUCT(OFFSET(PSI, 0, 0, 1, l), OFFSET(PSI, 0, b, 1, l)) as the function for the Table. Notice that custom number format code ';;;' is used in cell H15 to hide its value.

Figure 2 a is the Table of cumulative values of G. Cumulative of $g_{i,i}$'s are listed in the range F26:F35. The cumulative $g_{h,h+b}$'s are listed along the rows in the range G26:O34. We name Figure II-a (E26:O35) as "Cumulative_Table**,**" then create the G-Value Table to list the cumulative values of $g_{i,i}$'s and $g_{h,h+b}$'s by using the Data/Table command with
=IF(D42>D41, VLOOKUP(D42-D41, Cumulative_Table,D41+2,0),0) in cell F40, and G38 and G39 as row input cell and column input cell, respectively. The values of $g_{i,i}$'s and $g_{h,h+b}$'s are then used to find the variances for different lead time values. Given $\sigma_a = 5.78$, we entered =(G41+2*SUM(H41:N41))*5.78^2 in cell O41 to compute $Var[U_t(j)]$ for h =1. The variances for lead time values ranging from 2 to 8 can be obtained by extending the formula.

	E	F	G	H	I	J	K	L	M	N	O
23											
24	Figure II a - Cumulative - G										
25		0	1	2	3	4	5	6	7	8	9
26	1	1	0.79	1.8698	3.129276	4.500151	5.940094	7.422858	8.932172	10.45795	11.99393
27	2	2.6241	1.643042	3.717828	6.060295	8.568725	11.18005	13.85517	16.56985	19.30905	
28	3	5.414168	3.003024	6.558081	10.4554	14.56492	18.806	23.12865	27.50188		
29	4	9.790516	4.729608	10.09824	15.86306	21.87353	28.03629	34.29347			
30	5	16.04616	6.70359	14.1049	21.93881	30.04092	38.30932				
31	6	24.37524	8.838686	18.41333	28.44426	38.75809					
32	7	34.90291	11.07664	22.91365	35.22208						
33	8	47.70861	13.37952	27.5348							
34	9	62.8423	15.72308								
35	10	80.33523									

=IF(G39>G38,VLOOKUP(G39-G38,Cumulative_Table,G38+2,0),0)

	E	F	G	H	I	J	K	L	M	N	O
38		Row	0	Figure II-b. Excel Table for G-Value.							
39		Col	1	b-value							Variance
40		1	0	1	2	3	4	5	6	7	
41	Lead Time h	1	1	0	0	0	0	0	0	0	33.4084
42		2	2.6241	0.79	0	0	0	0	0	0	140.4523
43		3	5.4142	1.643	1.8698	0	0	0	0	0	415.5956
44		4	9.7905	3.003	3.7178	3.1293	0	0	0	0	985.2395
45		5	16.046	4.7296	6.5581	6.0603	4.5002	0	0	0	1995.899
46		6	24.375	6.7036	10.098	10.455	8.5687	5.9401	0	0	3605.011
47		7	34.903	8.8387	14.105	15.863	14.565	11.18	7.4229	0	5975.155
48		8	47.709	11.077	18.413	21.939	21.874	18.806	13.855	8.9322	9270.828

Figure 2 a. Cumulative – G Table, b. G-Values Table and Variance $Var[U_t(j)]$

6. Conclusion

In this paper, we employee the Box-Jenkins forecasting technique to deal with the cases when the period demands are not independent, but exhibits a serially correlated demand process which can be represented as an autoregressive-moving average – ARMA (p.d.q)-model for both deterministic and discrete stochastic lead time. We also present an Excel based methodology for finding the reorder point and the safety stock level. This approach is flexible and capable of handling the realistic scenario when both the demand and lead time are randomly distributed. Developing an Excel template does not require high-level programming knowledge and skills. In addition, the build-in probability density functions, distribution functions, and the Data Table command tremendously simplify the iterative computations by eliminating the need to look for values from the statistical tables. Moreover, the updating equations of Box-Jenkins together with the what-if analysis capability of Excel make it possible to update the reorder point and safety stock periodically. The user friendliness and built-in capabilities of Excel makes a spreadsheet inventory-control application a low-cost tool and model simulator which is easy for whatever modification necessary to better adapt to needs and environments of the market.

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