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Elemental Components in the Construction of a 4D Linear Programming Graphical Representation for Animation

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Abstract

The purpose of this study is to extend graphical representations of linear programming problems from two to four dimensions with animation. This involves the development of elemental components that can be integrated into an animation which constructs the feasible region and objective function as four dimensional objects in a four dimensional space. These components form the basis for an animation which, when developed in an incremental and logical manner, provide an intuitive understanding of the four dimensional graphical representation. The study focuses primarily on visual references in the construction of four dimensional objects, with the most complex object being a four dimensional rectangular parallelepiped. Reference to the geometric and mathematical properties of four dimensional objects is kept to a minimum.

The elemental components in this paper are the most basic for the development of slides which, when displayed in an orderly time sequence, provide a four dimensional animation of a 4D linear programming model which can be projected on a two dimensional plane. In this manner, the animation can be developed in Power Point and displayed using the Power Point pull down menu for "Slide Show."

This study includes higher order elemental components in the decomposition of a four dimensional rectangular parallelepiped. This is intended to enhance the visual skills of an observer in the transition of making observations from two to four dimensions.

1. Introduction

Two dimensional graphical representations of linear programming problems have long been a staple in texts on operations research and management science. Discussion of the two dimensional problem serves to illustrate linear programming concepts in a concrete manner by making analogies from two dimensional space and applying these analogies to higher dimensional spaces, as opposed to taking a purely abstract approach for linear programming problems with four or more decision variables. Such concepts include, *for example*, a two dimensional visual representation of the feasible region as a rectangle, where the corresponding feasible region in a four dimensional problem would be a four dimensional rectangular parallelepiped; a fixed slope objective function which can vary on a two dimensional graph until visually positioned at the optimum, corresponding to a four dimensional plane moving in a four dimensional space; a pattern of points as a feasible region on a two dimensional graph for an integer programming problem, corresponding to points consisting of four coordinates for a four dimensional integer programming program; and multiple optimum solutions when the optimum position of the objective function coincides with a line segment on the boundary of a two dimensional feasible region, corresponding to multiple optimum solutions from a three dimensional side of a four dimensional polygon coinciding with a four dimensional plane. Other concepts that are illustrated by the two dimensional graph follow in explaining the behavior of an algorithm for higher dimensional problems. For example, the simplex algorithm is suitable for higher dimensional problems and does so with iterations corresponding to adjacent vertices along the feasible region, with improvements in the value of the objective function until the optimum is achieved; and Gomory's cutting plane algorithm can be illustrated by adding additional constraints (cuts, which are actually lines and not planes when in two dimensions), until the integer optimum is found. In a problem with four decision variables, the former is analogous to points in the feasible region with each point being a vertex on a four dimensional polygon. The latter is analogous to having cuts that are four dimensional planes. The two dimensional graphical illustration offers these concrete visuals which, by analogy, can transcended to problems of higher dimensions. This allows concrete two dimensional visuals to be a reference in developing linear programming terminology for higher dimensional problems (e.g., terms such as cutting planes, n-dimensional feasible regions, and hyperplane objective functions), making the concepts underlying the terminology somewhat less abstract when applied to n-dimensional spaces.

There are numerous studies in the literature on n-dimensional spaces which include geometric representations that are applicable in showing four dimensional feasible regions for graphical representations of linear programming problems. The simplest representations are four dimensional cubes, sometimes referred to as tesseracts, hypercubes, or tetracubes. Examples of studies that focus on the mathematical and visual properties of tesseracts as subsets of polytopes (defined as n-dimensional solids bounded by n-1 dimensional enclosures) include Brisson [1], Coxeter [2], and Hilbert and Cohn-Vossen [3]. Less rigorous studies of higher dimensional solids include Bakst [4], Dewdney [5], and Turney [6]. These studies tend to be limited to tables and/or the derivation of simple formulas to determine geometric properties on n-dimensional solids such as the number of vertices, edges, faces, and volumes that enclose these solids as a function of dimensions (i.e., as a function of "n"). Similar studies of note can be found on the internet (e.g., [7]; [8]; [9]).

Leading studies of romantic accounts of higher dimensions tend to be based on Edwin A. Abott's *Flatland: A Romance of Many Dimensions* [10]. Taking Abott's basic theme of living beings in a two dimensional world to higher dimensions is the common characteristic in works by Pickover [11], Rucker [12], and Stewart [13]. However, such stories should not be taken lightly given that they provide a realistic account of the properties of a spatial world of higher dimensions, and enhance our understanding of higher dimensional geometry (sometimes referred to as hypergeometry [1]).

Two studies by Cosgrove develop four dimensional graphical representations of linear programming problems. The first provides a static graphical representation of a four dimensional rectangular parallelepiped as a feasible region enclosed by a four dimensional plane [14]. The second study presents a two dimensional linear programming problem with two constraints as a four dimensional graphical representation, where two additional spatial dimensions are added to represent slack variables in a demonstration of the iterations of the simplex algorithm [15].

The purpose of this study is to identify the key elemental components that can be integrated into an animation which constructs both a feasible region and objective function of a four dimensional linear programming problem. These components would form the basis for an animation which, when combined in a piecemeal and logical manner, would provide an intuitive understanding of the four dimensional graphical representation. Construction would consist of two four dimensional objects. The first would represent the feasible space determined by a set of constraints which defines a four dimensional rectangular parallelepiped. The second object would represent the objective function with four decision variables which defines a four dimensional plane (sometimes referred to as a hyperpland, a flune, or a tetrealm in tetraspace [16]).

The second section of this study constructs objects that are limited to a four dimensional plane and a four dimensional rectangular parallelepiped, each corresponding to an example of an objective function and a feasible region in a four dimensional space. The third section discusses the decomposition of the feasible region in terms of the three dimensional volumes that enclose a four dimensional space. These sections depend on basic two and three dimensional geometrical objects as the elemental components (i.e., building blocks) which, when properly sequenced, construct or decompose the four dimensional objects under discussion. This study focuses on visual representations and not on the mathematical properties of objects in four dimensional spaces.

2. Feasible Regions and Objective Functions for Two, Three, and Four Dimensional Formulations

Referring to a series of linear programming formulations in [14] which progressively add dimensions to the two and three dimensional formulations, consider the following two dimensional formulation as the starting point in this progression:

$$\begin{aligned} \text{Max } Z = 10x_1 + 8x_2 & \\ \text{s.t.} \quad x_1 \leq 2 & \\ x_2 \leq 1 & \\ x_1 \geq 0 & \\ x_2 \geq 0. & \end{aligned} \tag{1}$$

The graphical representation is illustrated in Figure 1, where the solution (point B) follows by moving the objective

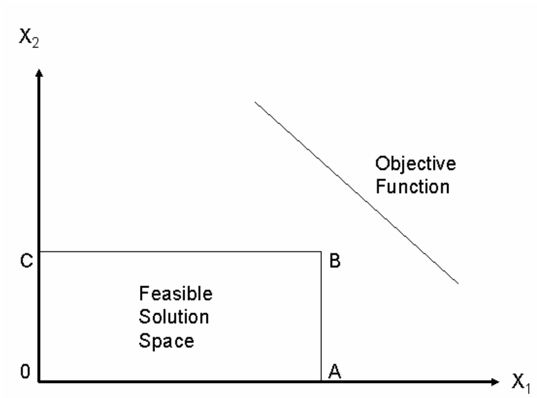

Fig. 1 2D Linear Programming Formulation

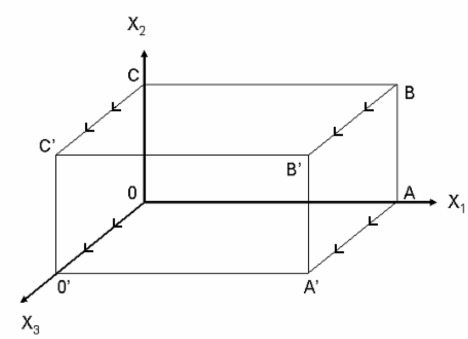

Fig. 2 3D Feasible Region

function such that it touches point B. Assume that another dimension is added to the problem leading to the following three dimensional problem:

$$\begin{aligned} \text{Max } Z = 10x_1 + 8x_2 + 9x_3 & \\ \text{s.t.} \quad x_1 \leq 2 & \\ x_2 \leq 1 & \\ x_3 \leq 1 & \\ x_1 \geq 0 & \\ x_2 \geq 0 & \\ x_3 \geq 0. & \end{aligned} \tag{2}$$

The feasible space for the three dimensional problem is shown in Figure 2. In this simple example, an animation can be developed to extend forward the two dimensional rectangle along the X_3 axis by stopping at $x_3 = 1$. The rectangle *OABC* serves as an elemental component in the animation by sweeping out a thee dimensional volume in the direction shown by the arrows (out from the plane of the paper), leading to the rectangular parallelepiped *OABCO'A'B'C'*. If Z = 72 is arbitrarily selected with the intention of the objective function being beyond the feasible space (as in the two dimensional representation in Figure 1), the intercepts along X_1, X_2, and X_3 have coordinates (7.2, 0, 0), (0, 9, 0), and (0, 0, 8). Extending the three axes in Figure 2 to accommodate these intercepts followed by connecting the intercepts leads to the three dimensional graphical representation in Figure 3, which includes the objective function as a plane and the feasible solution as the rectangular parallelepiped from Figure 2. Note that the line segments connecting the intercepts would be the elemental components for the animation of the plane.

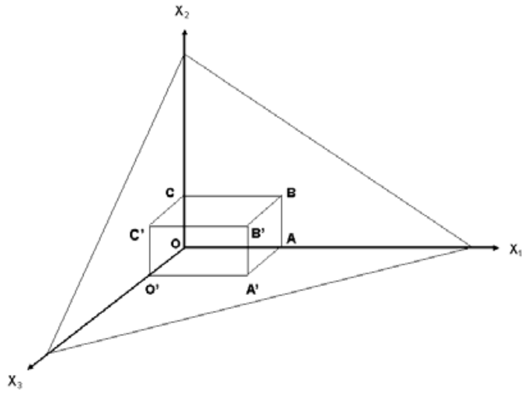

Fig. 3 3D Linear Programming Formulation

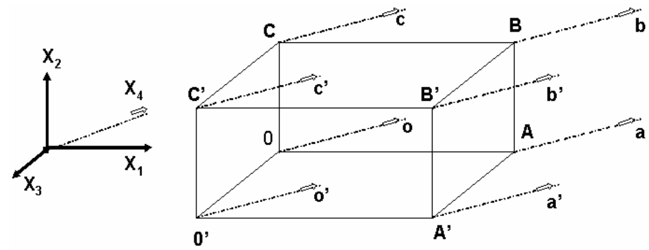

Fig. 4 Transition from 3D to 4D Feasible Region

The transition from the three dimensional to four dimensional formulation is based on the following four dimensional problem:

$$
\begin{aligned}
\text{Max } Z &= 10x_1 + 8x_2 + 9x_3 + 7x_4 \\
\text{s.t.} \quad & x_1 \leq 2 \\
& x_2 \leq 1 \\
& x_3 \leq 1 \\
& x_4 \leq 1 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0. \\
& x_4 \geq 0.
\end{aligned}
\tag{3}
$$

Figure 4 makes the transition into the fourth dimension by treating the three dimensional feasible region from Figure 2 as an elemental component, and pushes the entire three dimensional rectangular parallelepiped, *OABCO'A'B'C'*, back behind the plane of Figure 2 as shown in Figure 4. This push corresponds to a direction in the fourth dimension, with a coordinate system reference which indicates that the new direction is along the positive direction of the X_4 axis. It is now a simple operation to connect the lower case points (*oabco'a'b'c'*) in Figure 4 to develop the feasible region represented by the four dimensional linear programming formulation in (3). The result is shown in Figure 5.

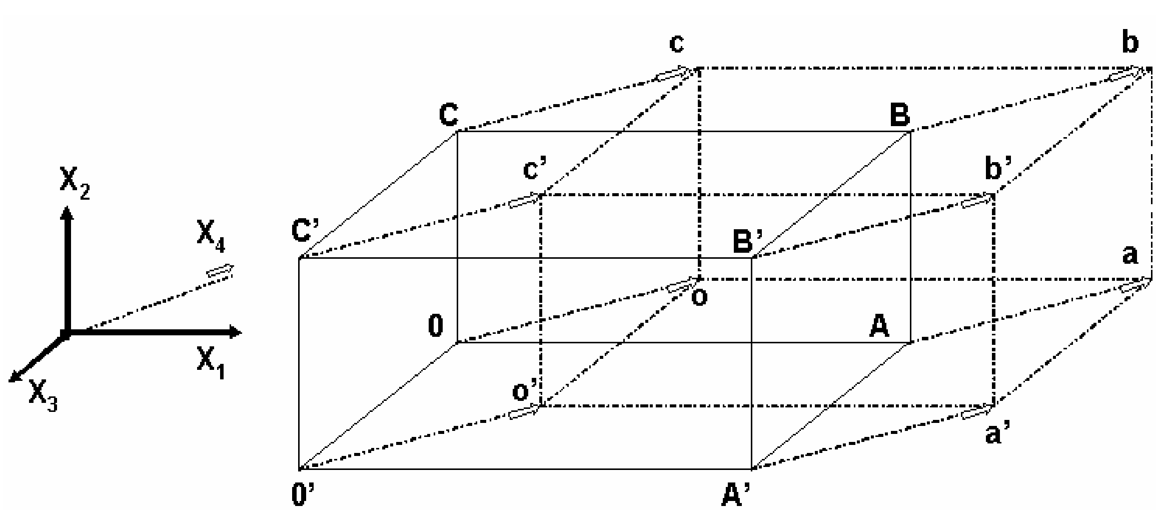

Fig. 5 4D Feasible Region

An animation leading to Figure 5 follows by taking the elemental component (i.e., *OABCO'A'B'C'*) and gradually stretching it out along the line segment *Oo* in the positive direction of the X_4 axis. This "stretch" is easily done in Power Point by developing a series of slides to incrementally perform the stretch, and then sequencing the slides in a time lapse controlled by features in the "Slide Show" pull down menu.

If Z = 108 is arbitrarily selected for the objective function in (3) with the intention of the objective function being beyond the feasible space (as in the two dimensional representation in Figure 1), the intercepts for X_1, X_2, X_3 and X_4 have coordinates (10.8, 0, 0, 0), (0, 13.5, 0, 0), (0, 0, 12, 0), and (0, 0, 0, 15.4). These intercepts correspond to the points T, V, U, and W in Figure 6. Again, the fourth dimension is represented by the X_4 axis with the positive

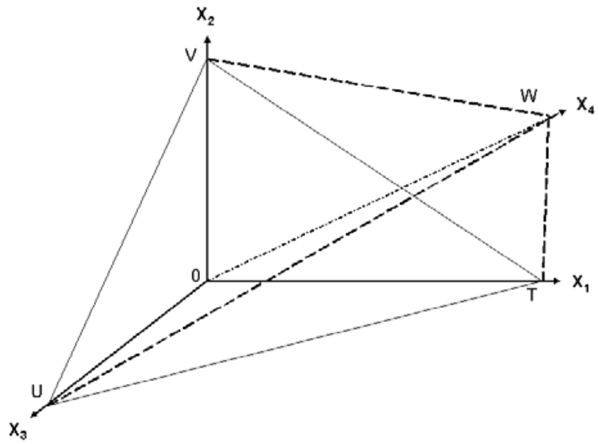

Fig. 6 4D Plane (Hyperplane)

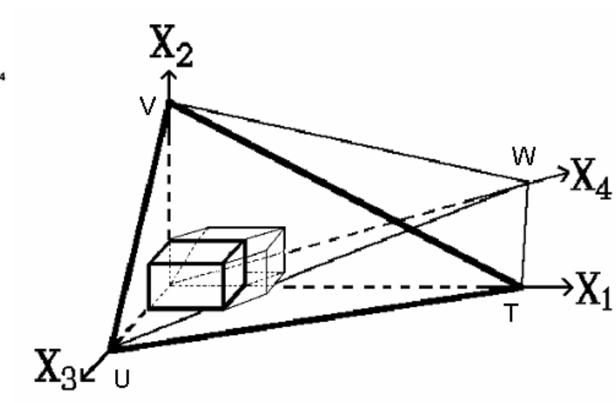

Fig. 7 4D Linear Programming Formulation

direction into the plane of the figure. For an animation, the elemental components can be defined as the triangles *TVU*, *TVW*, *TUW*, and *VUW*. A series of four transparent overlapping slides, each containing one of the triangles, leads to the four dimensional representation in Figure 6. Combining Figure 5 and Figure 6 leads to Figure 7, the four dimensional graphical representation of the four dimensional linear programming problem as specified in (3).

3. Decomposition of a Four Dimensional Feasible Region

A simple decomposition of the three dimensional region from Figure 2 is illustrated in Figure 8 in terms of four rectangles and two squares. The analogy of performing such a task is much more involved in a four dimensional

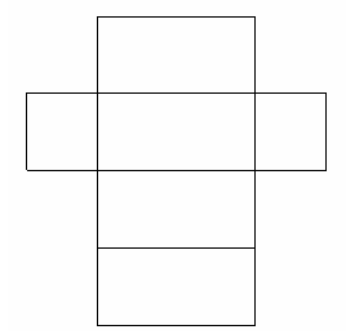

Fig. 8 Decomposition of 3D Feasible Region

space for the feasible region described in Figure 5. This follows in that Figure 2 has a volume that is enclosed by simple two dimensional rectangles and squares, while Figure 5 has a volume that is enclosed by three dimensional rectangular parallelepipeds and cubes. Achieving a meaningful understanding of a decomposition in four dimensions requires that the enclosures (or sides) of the four dimensional volume be clearly identified.

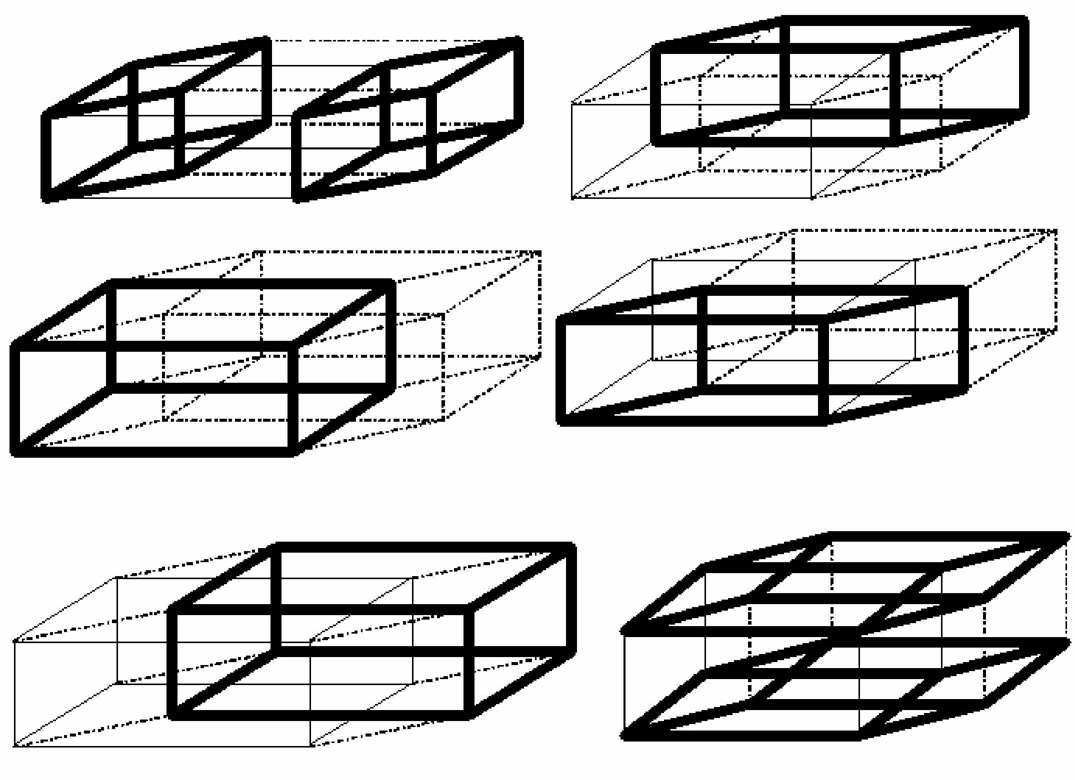

Fig. 9 Enclosures (Sides) for 4D Feasible Region

A careful and detailed examination of Figure 5 reveals eight three dimensional enclosures, each of which is specified once in Figure 9 and twice in Figure 10. Note that Figure 10 shows the enclosures separated on the top and attached on the bottom, where the latter is an analogy to the three dimensional example in Figure 8. Two enclosures

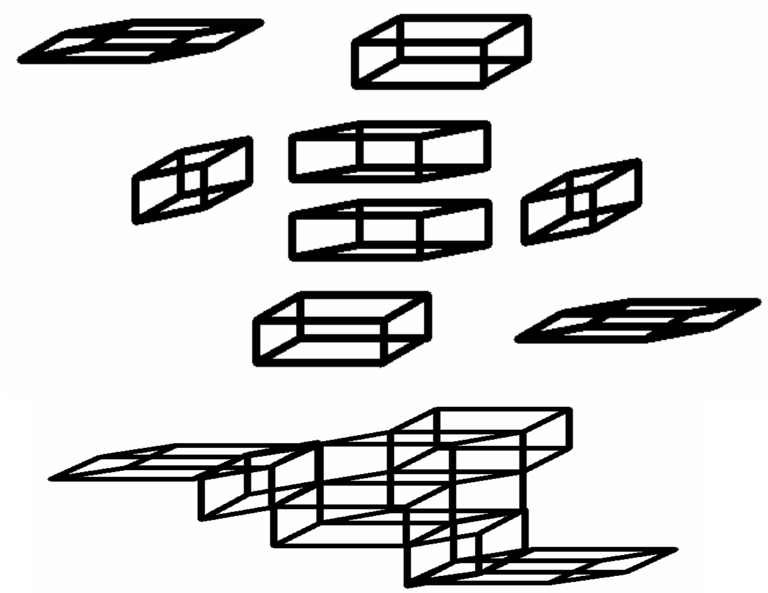

Fig. 10 Two Equivalent Decompositions of the 4D Feasible Region, with Enclosures Separated and Grouped

are cubes and the other six are rectangular parallelepipeds. The primary intent of an animation of a four dimensional decomposition is to help observers make more informed and detailed observations of these four dimensional objects.

4. Conclusion

Interest in four dimensional graphics has been enhanced with the availability of low cost graphics software packages and improved resolutions with computer monitors, permitting more detailed and intricate graphical constructions. With the advent of the concept of "data mining," which offers a new source of higher dimensional data from government and corporate data bases, it is expected that a greater emphasis on research for the development of tools for four (or more) dimensional representations will follow [17]. In general, as graphics software further develops to accommodate geometric representations in three and four spatial dimensions, visual searches for correlations among the data will be available to supplement tabular data of correlation measurements. For linear programming in particular, it seems reasonable to speculate that higher dimensional graphical representations may assist in the development of new approaches for solving linear and integer programming problems more efficiently than conventional approaches. However, setting all speculation aside, the graphical representations in this study have a pedagogical value in that they offer an enhanced understanding of graphical linear programming beyond that of the usual two dimensional problem.

A logical extension of this study is the development of the first graphical representations for transportation and assignment problems which require a minimum of four decision variables. This is currently in progress by the author.

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