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Practical Technique for Determining Minimum Variance Portfolio

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Abstract

In asset allocation finding the investment proportions that minimize portfolio risk, or determine the Minimum Variance Portfolio (MVP), is an important step toward prudent portfolio management. This work finds these optimal proportions introducing a new technique that is different from an ordinary constraint optimization method. Such technique computes the optimal investment proportions for each asset and simultaneously calculates the numerical value of the minimum variance portfolio without the need to take the first and second derivatives and provides solutions without solving systems of linear equations.

Since statistical quantities for each asset are quantified in terms of their expected rate of return and variance of returns, using this method along with modern portfolio theory portfolio managers can assess the risk-reward tradeoffs and compare-contrast the characteristics of different portfolios in asset allocations.

Background

The work of Markowitz, Sharpe, Fama, Treynor and Black defined the current modern portfolio theory in finance literature. From a given collection of m risky assets and n constraints, the optimal weights can be found by solving a system of $m+n$ linear equations for $m+n$ unknowns. The optimization process uses statistics to examine the market as a whole and search for portfolios with desired performance characteristics. Mathematically, this problem is presented in a form of an objective function with m variables and n linear constraints. In addition to m risky assets, we assume there is available a risk-free asset with a fixed rate of return, R_f. Expected returns of assets in the portfolio are determined through the use of historical data, and risk is measured through the use of the standard deviation.

The model for analyzing risk focuses on probability distributions of some quantifiable outcome. Since an investment's rate of return is the relevant outcome of an investment, financial risk analysis focuses on the probability distribution of rates of return. Modern portfolio theory uses the mean and variance of the returns as a basis of investment decisions in the risk-reward space. The logic of using only the rate of return may seem simplistic compared with more in-depth security analysis techniques that stress ratio analysis of financial statements, management interviews, industry forecasts, the economic outlook, and financial markets. In practice, however, there is no contradiction between these two approaches. After the fundamental security analysis is complete, one needs to convert the estimates into several possible rates of return and attach probability estimates to each. The security analyst's consideration of the market demand for the firm's products, firm's success in research and development, management depth and ability, and macroeconomic conditions are all duly reflected in the forecasted rates of return and their probabilities. Thus, the variability of the expected return is a measure of risk grounded in fundamental analysis of the firm, its industry, and the economic outlook.

Discussion

A central component to the calculation of efficient portfolios is the variance-covariance matrix. Let the variance-covariance matrix for an *m*-asset portfolio be denoted as:

$$V = \begin{bmatrix} Cov(r_1,r_1) & Cov(r_1,r_2) & Cov(r_1,r_3) & \cdot & Cov(r_1,r_m) \\ Cov(r_2,r_1) & Cov(r_2,r_2) & Cov(r_2,r_3) & \cdot & Cov(r_2,r_m) \\ Cov(r_3,r_1) & Cov(r_3,r_2) & Cov(r_3,r_3) & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ Cov(r_m,r_1) & Cov(r_m,r_2) & \cdot & \cdot & Cov(r_m,r_m) \end{bmatrix}$$

By recognizing that the covariance of one asset to itself is the variance and that the covariance between two assets is constant irrespective of the order, the variance-covariance matrix is symmetric and can be simplified as following:

$$V = \begin{bmatrix} \sigma^2(r_1) & Cov(r_1,r_2) & Cov(r_1,r_3) & \cdot & Cov(r_1,r_m) \\ & \sigma^2(r_2) & Cov(r_2,r_3) & \cdot & Cov(r_2,r_m) \\ & & \sigma^2(r_3) & & \cdot \\ & & & \cdot & \cdot \\ & & & & \sigma^2(r_m) \end{bmatrix}$$

Also let x denote a vector of portfolio weights for the m assets.

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & \cdot & \cdot & x_m \end{bmatrix}$$

The variance of the portfolio can be computed using the following equation:

$$\sigma^2(r_p) = xVx^T$$

The equations above hold true for a portfolio without any constraints. However, in practical applications, constraints are a necessity. In particular, the two constraints of interest are: the sum of portfolio weights equals 1.0 and the computed portfolio return equals to the required rate of return. A Lagrangian function can be defined to incorporate these two constraints:

$$L = \sigma^2(r_p) - \lambda_1(x_1 + x_2 + x_3 + ... + x_m - 1) - \lambda_2(x_1k_1 + x_2k_2 + x_3k_3 + ... + x_mk_m - R_p) \qquad (1)$$

where λ_i are Lagrangian multipliers, k_i are the average rates of return for the i^{th} asset, and R_p is the required rate of return.

Differentiating the Lagrangian function with respect to the portfolio weights and Lagrangian multipliers and setting the result equal to zero will produce an equation that leads to portfolio weights that minimize the function. The resulting equation can be written in the form of the linear algebraic equation:

$$Hy = z \qquad (2)$$

where H is a bordered Hessian matrix as denoted below:

$$\begin{bmatrix} 2\sigma^2(r_1) & 2Cov(r_1,r_2) & 2Cov(r_1,r_3) & \cdot & \cdot & 1 & k_1 \\ 2Cov(r_2,r_1) & 2\sigma^2(r_2) & 2Cov(r_2,r_3) & \cdot & \cdot & 1 & k_2 \\ 2Cov(r_3,r_1) & 2Cov(r_3,r_2) & 2\sigma^2(r_3) & \cdot & \cdot & 1 & k_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & 0 & 0 \\ k_1 & k_2 & k_3 & \cdot & \cdot & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x1 \\ x2 \\ x3 \\ . \\ . \\ -\lambda 1 \\ -\lambda 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 1 \\ Rp \end{bmatrix}$$

y is the vector of portfolio weights and Lagrangian multipliers:

$$y^T = \begin{bmatrix} x_1 & x_2 & x_3 & \cdot & \cdot & -\lambda_1 & -\lambda_2 \end{bmatrix}$$

z is defined as follows:

$$z^T = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & 1 & R_p \end{bmatrix}$$

Solving for the vector y produces the portfolio weights that are necessary to achieve the desired portfolio rate of return, R_p.

$$y = H^{-1} z \tag{3}$$

Previously, the determination of an efficient portfolio involved graphical techniques. For example, starting with the minimum variance set diagram, one would locate a position on the vertical axis equal to the risk-free rate, R_f. Using this point as a basis, the Capital Market Line (CML) is constructed as a tangent to the efficient frontier. The line itself has the property of the greatest possible slope.

The tangent point results in an efficient portfolio standard deviation of which is estimated from the slope of the CML and the portfolio rate of return. This method produces an estimated portfolio risk accuracy of which depends on the density of the discrete sample points.

In the new approach presented below, the exact value of the risk, i.e. standard deviation, can be attained by utilizing subcomponents of the bordered Hessian matrix, the Lagrangian multipliers, and the constraint values. In examining the bordered Hessian matrix, in equation (3), it is noted that the upper left terms are consistent with the variance-covariance matrix, *V*. Additionally, due to the symmetric property, the upper right and lower left elements are equivalent. Thus, the bordered Hessian matrix can be sub-divided into four sub-matrices as shown:

$$H = \begin{bmatrix} 2\sigma^2(r_1) & 2Cov(r_1,r_2) & 2Cov(r_1,r_3) & \cdot & \cdot & 1 & k_1 \\ 2Cov(r_2,r_1) & 2\sigma^2(r_2) & 2Cov(r_2,r_3) & \cdot & \cdot & 1 & k_2 \\ 2Cov(r_3,r_1) & 2Cov(r_3,r_2) & 2\sigma^2(r_3) & \cdot & \cdot & 1 & k_3 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & 0 & 0 \\ k_1 & k_2 & k_3 & \cdot & \cdot & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2V & K \\ K^T & 0 \end{bmatrix}$$

where V is the m x m variance-covariance matrix and K is a m x n matrix as shown below (m is the number of assets and n is the number of constraints):

$$V = \begin{bmatrix} \sigma^2(r_1) & Cov(r_1,r_2) & Cov(r_1,r_3) & \cdot & Cov(r_1,r_m) \\ & \sigma^2(r_2) & Cov(r_2,r_3) & \cdot & Cov(r_2,r_m) \\ & & \sigma^2(r_3) & & \cdot \\ & & & \cdot & \cdot \\ & & & & \sigma^2(r_m) \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & k_1 \\ 1 & k_2 \\ 1 & k_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & k_m \end{bmatrix}$$

Additionally, the vector y can be divided into two sub-vectors: the portfolio weights, x, and the Lagrangian multipliers, λ. This is shown below.

$$y^T = \begin{bmatrix} x_1 & x_2 & x_3 & \cdot & \cdot & -\lambda_1 & -\lambda_2 \end{bmatrix} \qquad .$$

$$= \begin{bmatrix} \underline{x} & \underline{\lambda} \end{bmatrix}$$

Finally, the vector z can also be divided into two sub-vectors: a zero vector and the constraint values, C, as shown below.

$$z^T = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & 1 & R_p \end{bmatrix}$$

$$= \begin{bmatrix} \underline{0} & \underline{C} \end{bmatrix}$$

With the above sub-matrices and sub-vectors defined, equation (3) can be rewritten as follows:

$$Hy = z$$

$$\begin{bmatrix} 2V & K \\ K^T & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{C} \end{bmatrix}$$

The equation can now be separated into two distinct matrix operations. The result is the following:

$$2V\underline{x} + K\underline{\lambda} = 0$$

$$K^T\underline{x} = \underline{C}$$

By noting that the relationship $\underline{x}^T K = (K^T\underline{x})^T$ holds true, the first set of equations can be used to determine the portfolio standard deviation.

$$\begin{aligned} 2V\underline{x} + K\underline{\lambda} &= 0 \\ 2\underline{x}^T V\underline{x} + \underline{x}^T K\underline{\lambda} &= 0 \\ 2\underline{x}^T V\underline{x} + \underline{C}^T\underline{\lambda} &= 0 \end{aligned}$$

Thus the portfolio variance and standard deviation are given by the following equations:

$$\sigma^2(r_p) = \underline{x}^T V \underline{x}$$
$$= -\frac{1}{2}\underline{C}^T \underline{\lambda}$$

$$\sigma(r_p) = \sqrt{-\frac{1}{2}\underline{C}^T \underline{\lambda}}$$

An important application of this method is to find minimum variance portfolio where the sum of portfolio weights equal 1.0. The border Hession matrix simplifies to variance-covariance matrix with one border as: $Hy = z$

$$\begin{bmatrix} \sigma^2(r_1) & Cov(r_1,r_2) & Cov(r_1,r_3) & \cdot & \cdot & 1 \\ Cov(r_2,r_1) & \sigma^2(r_2) & Cov(r_2,r_3) & \cdot & \cdot & 1 \\ Cov(r_3,r_1) & Cov(r_3,r_2) & \sigma^2(r_3) & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & 0 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \\ -Var(p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix}$$

That means the last column of the $[H]^{-1}$ provides the proportions that ought to be invested in each asset and the last number in that column is the numerical value of the minimum variance portfolio. Thus, it generates the results without derivatives or solving the systems of linear equations.

Conclusions

This work offers a method to make the process of constraint optimization in asset allocation easier and more straightforward. For practitioners, as well as academicians, the above method makes the formation of portfolios with different desired rates of return possible by using one simple calculation. In addition, the computation of risk (standard deviation) of such portfolios is simple and far more accurate than traditional techniques.

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