A Benchmark of Finding the Criteria for Optimal Portfolio Choice

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Abstract

In today's rapidly changing economic environments, global portfolio investors are invariably confronted with dynamic and fluid choice of criteria that enable them to exercise subjective judgment on asset portfolio choice decisions. The subjective investment portfolio choice includes the discretion to purchase some assets that do not meet its normal investment criteria, when they perceive unusual information which incurs event jumps in the asset return. This paper studies on what criterion or criteria should be used when the asset in the investment portfolio is subject to event jumps as well as how the criterion or criteria can be found. We use marginal excess return in the value add return function for indicating the return decreasing under information impact and finding a critical value related to the growth optimum portfolio choice on a risk adjusted basis. We reveal, as a special case, the optimal portfolio is equivalent to the benchmark portfolio, if the investor is myopic of constant relative risk aversion.

1. Introduction

Managers of the large multinational firms and investment funds usually seek the highest total portfolio return over time that is consistent with emphases on both capital appreciation and investment income/return under a specified risk level. They pursue its objective by investing in a diversified portfolio and possess the discretion to purchase some assets that do not meet its normal investment criteria, when they perceive an unusual information impact on the investment. As investors hold different portfolios, partially reflecting their different information shock, less informed investors/managers tend to propose more than the aggregate portfolio in assets about which they are more optimistic than the more informed investors. Suffering the information shocks has caused some forms of event jump on the investment portfolios. Since the less informed investors/managers do not know the exact structure of the portfolio of aggregate risk, the deviation from this portfolio can be interpreted as a form of estimation risk. In the past (e.g., Kandel and Stambaugh [1996]), estimation risk has been studied under homogeneous information, in which case it only adds to asset return variance.

Often active portfolio managers/investors aim to beat the performance of a selected target benchmark, whereas passive portfolio managers/investors just track the selected benchmark. The most general setting is one in which the investment opportunity set depends on a set of unobservable state variables that evolved estimation risk arising from the unknown nature of both the parameters and the state variables. Hence the process also involves learning. In the simplest setting, Brennan (1998) evaluated the role of learning in the dynamic portfolio allocation decision, with the assumption that the investor is able to trade continuously and the value of the risky asset follows a diffusion process. Brennan assumed that the investor learns about the mean return in the simple dynamic portfolio problem grounded in constant investment opportunity set but is uncertain about the mean return on the risky asset. The problem of finding an optimal estimation risk or mean asset return can be logically separated from the problem of finding the decision rules (see Gennotte (1986)). We study how the adjustment of the portfolio achieves a positive premium depends on the growth optimal portfolio which can be viewed as a reference portfolio. Under the framework of mean return and its variability, we find the implementation of the criteria for adjusting practical investment portfolio choice.

Thus the necessity for introducing more specialized instruments is driven by an inherent incompleteness of the existing markets, which forces investors to seek out new securities. Empirical studies have documented that incompleteness may derive from the fact that asset prices have jumps and volatility is stochastic. Such models, for example, Platen (2004) and Christensen & Platen (2005), investigate the structure of market prices of risk as markets become approximately complete. Their models consider also the limits of traded securities, characterizing explicitly the growth optimum portfolio and diversification in such markets.

In this paper, we extend the work on the role of estimation risk in a single period context of Kandel and Stambaugh (1996) and the analysis by Browne (1999) on active portfolio management problems with objectives related to the achievement of relative performance goals and shortfalls. Kandel and Stambaugh (1996) consider the effect of estimation errors in the return prediction equation on the optimal portfolio strategy in a multi period setting, an important aspect of the dynamic problem that is induced by learning. Browne (1999) considered a general problem in an incomplete market where the benchmark was only partially correlated with the active investor’s investment opportunities. In Brown’s (1999) study, the investment objectives are related to the achievement of investment goals and shortfalls relative to the benchmark. We observe that if the proportion associated with an asset class is positive, the portfolio rebalancing requires selling an asset when its price rises relative to the other prices, and conversely, buying the asset when its price drops relative to the others.

The application of portfolio theory was originally developed to analyze the problem of an investor who faced a single period horizon, and a known investment opportunity set. It was extended by Merton (1971), Breeden (1979) and others to allow for a multi-period horizon in which investment opportunities might either be constant, time-dependent, or even stochastic. Brennan and Schwartz (1996) have implemented the Merton-Breeden continuous time approach to dynamic portfolio planning by specifying state variables that describe the state of the economy and estimated their dynamics. These studies are lack of considering the stochastic process parameters whose values can only be estimates. We propose an information uncertainty setting, the use of the growth optimum portfolio as numeraire, implying the associated martingale measure is the historical measure itself. Hence, particularly for event driven uncertainty, this may have an advantage, since default intensities and jump sizes, observed under this measure, are otherwise difficult to calibrate. We aim to study the specific objectives that were explicitly solved for multi-objective cost functions: a) maximizing the probability that the investor’s wealth achieves a certain performance goal relative to the benchmark before falling below it to a predetermined shortfall; b) minimizing the expected time to reach the performance goal; c) maximizing the expected reward obtained upon reaching the goal; and d) minimizing the expected penalty paid upon falling to the shortfall level. The corresponding optimal policies obtained there are all constant proportions, or in constant mix, portfolio allocation strategies, whereby the portfolio is continuously rebalanced so as to always keep a constant proportion of wealth in the various asset classes, regardless of the level of instant wealth.

Our use of marginal excess return in the value add return function for indicating the return decreasing under information impact and finding a critical value related to the growth optimum portfolio choice on a risk adjusted basis. In this framework we obtain a closed form solution in an incomplete market with jumps and show how to use this framework for modeling diversified portfolios.

The remainder of the paper is organized as follows. The next section presents the modeling of the dynamic portfolio decision, reflecting the differences in equity risk, as measured by the volatilities of stock return. Section 3 establishes the optimal portfolio and provides analytical solutions to the optimal portfolio choice problem for events jump. Section 4 finds the solution to the generalized optimal portfolio return to set up the measurement of risk-adjusting and criteria for portfolio. Section 5 provides empirical results and Section 6 summarizes the results and makes concluding remarks.

2. Modeling the Dynamic Portfolio Decision in Incomplete Market

Assume that a fund manager or an investor who knows the diffusion parameter σ, the optimal estimates about the mean return, represented by m, we can formulate the portfolio optimization problems for the fully informed agent from the maximization of Merton proportion

$$\phi_t = \frac{m-r}{\gamma \cdot \sigma^2} \tag{1}$$

where γ is the constant relative risk aversion coefficient, σ is the instantaneous standard deviation of diffusive asset return, and compound rate of asset which is not random, r. At time zero the fund manager/investor views the distribution of asset return, μ as normal with mean m_0 and variance v_0. Similarly, denote the expectation and variance of mean return at time *t* by m_t and v_t respectively. Following the work of Liptser and Shiryayev (1978), change in the conditional variance v_t at time t, is determined by v_0 and can be obtained using the differential equation:

$$dv = \left[-\frac{v^2}{\sigma^2} \right] dt \tag{2a}$$

Equation (2a) can be rewritten as

$$v_t = \frac{v_0 \sigma^2}{v_0 t + \sigma^2} \tag{2b}$$

This problem can be solved analytically by first guess of the value function followed by using the principle of optimality, which leads to the Hamilton-Jacobi-Bellman equation. In doing so, we arrive at the following expression for optimal weight on the risky asset by using the Cox and Huang (1989) method.

$$\phi_t = \frac{m - r}{\gamma \cdot \sigma^2} + \phi_{ht} \tag{3}$$

where ϕ_{ht} represents the fully informed agent's need to hedge against the realization of the estimated mean. Assuming ($m - r$)>0, we can see that the hedging demand ϕ_{ht} is always negative for a conservative agent, while it might be positive for an aggressive agent. For a logarithmic agent, the hedging demand is zero.

In our setting, firstly the desire to hedge against changes in the assessed mean of the process depends on the agent's degree of risk aversion. The value function which arises from observation of the instantaneous realized return on the asset may lead the agent to invest more or less in the risky asset than he would under complete information. Next, we solve for the optimal portfolio strategy ϕ^* by conjecturing that the indirect utility function is of the form

$$J(W,t) = \frac{1}{1-\gamma} W^{1-\gamma} \exp(a(t)) \tag{4}$$

where W represents the wealth under the state m_t, $a(t)$ is a time dependent only function. Thus the value function J has two separable components: a time dependent and a wealth dependent. By Bellman's Principle of optimality, we obtain the Hamilton-Jacobi-Bellman equation as follows.

$$\max_{\phi_t}\{J_t + (r + \phi_t(m-r)W_t J_W + \frac{1}{2}\sigma^2\phi_t^2 W_t J_{WW} + \frac{1}{2}\frac{v_t^2}{\sigma^2} J_{mm} + v_t\phi_t W_t J_{mW}\} = 0. \tag{5}$$

Solving equation (5), we obtain the optimal portfolio

$$\phi^* = \frac{m \ -r}{\gamma \cdot \sigma^2} + \frac{v_t}{\gamma \cdot \sigma^2}. \tag{6}$$

In Equation (6) the first term is the Merton portfolio, and the second term is the hedge demand portfolio. Equation (6) also implies the following six relationships among the parameters. They are:

1. The coefficient of relative risk aversion γ rises as the optimal portfolio weight falls.
2. For the conservative investor, the optimal portfolio weight decreases as the time horizon increases, and for the aggressive investor, the optimal portfolio weight increases as variance of the private signal increases.
3. For the conservative investor, the optimal portfolio weight decreases as the variance of the private signal of more informed agent increases; and for the aggressive investor, the optimal portfolio weight increases as the time horizon increases.
4. The optimal portfolio weight falls as the variance of the asset return increases.
5. For the conservative investor, the optimal portfolio weight decreases as the variance of the earlier, increases; and for the aggressive investor, the optimal portfolio weight will increase as the variance of the earlier increases.
6. As time horizon becomes infinite, the optimal portfolio weight goes to zero. Although the different agents start at different points, they should follow somewhat similar paths, and eventually end up at approximately the same point. Since the agents with more information want to minimize the variance of their mean return estimates, they put relatively more weight on a signal with relatively lower variance. As time goes by, the agents learn more and more about the true mean and so their mean return estimates eventually converge to the true mean.

3. Portfolio Choice with Event Jumps

By applying Taylor expansion to the Euler equation and approximate the portfolio choice, we can extend our last formulation in equation (6) to *n* risky assets portfolio decision with information asymmetry. We assume that in the economy, there are n+1 assets: a riskless asset, treasury bond, represented by B, paying a constant rate of interest, r, and n risky assets whose prices $\mathbf{S}(t)$ =[$S_1(t)$,..., $S_n(t)]'$ are subject to information filtration. In continuous time finance, a popular way to generate discontinuity is to apply the compound Poisson Jump Model of Press (1967). Therefore, the return of risky assets follows a jump-diffusion process described by

$$\frac{d\mathbf{S}}{\mathbf{S}} = (\boldsymbol{\mu} - \boldsymbol{\lambda}\boldsymbol{\mu}_x)dt + \boldsymbol{\sigma} d\mathbf{Z} + \mathbf{X}dq \tag{7}$$

where $\boldsymbol{\mu} = [\mu_1, ..., \mu_n]'$ is the vector of mean return of n industrial assets, $d\mathbf{Z} = [dZ_1, .., dZ_n]' \sim N(0,1)$ is the vector of standard Brownian motion and "$'$" denotes the transposition operator. The $n \times n$ variance-covariance matrix $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$ is deterministic of diffusive returns. The variables in vector $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}'$ are random price-jumps size with means $\boldsymbol{\mu}_x = \begin{bmatrix} \mu_{x1} & \cdots & \mu_{xn} \end{bmatrix}'$. We define dq as a Poisson process with stochastic arrival intensity λ, i.e. $\Pr(dq = 1) = \lambda dt$. The term $\mu_x = E(X)$ captures the mean percentage jump in the asset price conditional on one jump occurrence. The jump size X_i, i=1,...,n, is assumed to be itself a random variable with a distribution $\ln X \sim N\left(\mu_x, \sigma_x^2\right)$. If μ, σ^2, μ_x, σ_x^2 and λ are constants, then

$$\ln \frac{S(T)}{S(t)} \sim \sum \frac{e^{-\lambda\tau}\left(\lambda\tau\right)^n}{n!} N\left(\mu\tau + n\mu_x, \sigma^2\tau + n\sigma_x^2\right) \tag{8}$$

where $\tau = T - t$, the log return distribution is described as a Poisson mixture of normal distribution. Using the relationship between cumulants and moments (Kendall and Stuart, 1963), it can be proven that this distribution is leptokurtic and therefore might better describe the actual stock price return behavior than the pure lognormal model given in the Appendix.

Now, we turn to the portfolio decision by applying the Taylor expansion to the Euler equation and approximating the portfolio choice. Maximizing the expected utility of the terminal wealth W_T, i.e. $\max_{\phi} E[U(W_T)]$, and the return of wealth process satisfies the self-financing condition. We obtain the following equation:

$$\frac{dR}{R} = \frac{dB}{B}(1 - \boldsymbol{\Phi}' \cdot \mathbf{1}) + \boldsymbol{\Phi}'(\frac{dS}{S}) = [r + \boldsymbol{\Phi}'(\boldsymbol{\mu} - r \cdot \mathbf{1} - \boldsymbol{\lambda}\boldsymbol{\mu}_x)]dt + \boldsymbol{\Phi}'\boldsymbol{\sigma} d\mathbf{Z} + \boldsymbol{\Phi}'\mathbf{X}dq, \tag{9}$$

where $\boldsymbol{\Phi} = [\phi_1, ..., \phi_n]'$ is a vector of a portfolio whose components sum to 1 and which indicates the investor's portfolio choice among the available investment opportunities and $\mathbf{1} = (1,1,...,1)'_{n\times 1}$, such that $\boldsymbol{\Phi}'\mathbf{1} = 1$.

4. Dynamic Active Portfolio Management

In general, portfolio managers' performance is typically judged by the return on their actively managed portfolio relative to the return of the preset benchmark. We focus on the determination of an investment strategy that is optimal relative to the performance of a benchmark portfolio for an investment criterion.

4.1 Benchmark Portfolio

Let $\mathbf{\Phi}_M = [\phi_{M1},...,\phi_{Mn}]'$ be the benchmark portfolio with ϕ_{Mi} being the fraction of the benchmark portfolio in the *i*th security. In solving for the optimal portfolio strategy, we adopt the standard stochastic control approach without any jumps. We solve for the benchmark optimal portfolio strategy $\mathbf{\Phi}_M^*$ without any jumps by first conjecturing that the indirect utility function is of the form

$$J(W,t) = \frac{1}{1-\gamma} W^{1-\gamma} \exp(A(t)), \tag{10}$$

where $A(t)$ is a time dependent only function. By the principle of optimal stochastic control leads to the Hamilton-Jacobi-Bellman (HJB) equation for the indirect utility function J. Given this function form, we take derivatives of $J(W,t)$ with respect to its arguments, $J_W = W^{-\gamma} e^A$, $J_{WW} = -\gamma W^{-\gamma-1} e^A$, substitute it into the HJB equation then we have:

$$\max_{\mathbf{\Phi}_M} \{J_t + W^{-\gamma}[r + \mathbf{\Phi}_M^{'}(\boldsymbol{\mu} - r \cdot \mathbf{1})]e^A dt + \frac{1}{2} e^A \cdot (-\gamma W^{-\gamma-1}) \mathbf{\Phi}_M^{'} (\boldsymbol{\sigma}\boldsymbol{\sigma}') \mathbf{\Phi}_M \; dt\} = 0 \tag{11}$$

Differentiate (11) with respect to the portfolio weight of the risky asset, $\mathbf{\Phi}_M$, and divide by $W^{-\gamma} e^A dt$ to obtain

the first-order condition: $(\boldsymbol{\mu} - r \cdot \mathbf{1}) - \gamma(\boldsymbol{\sigma}\boldsymbol{\sigma}')\mathbf{\Phi}_M^* = 0$. Hence the benchmark optimal portfolio weight can be

expressed by

$$\mathbf{\Phi}_M^* = (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1} \frac{\boldsymbol{\mu} - r \cdot \mathbf{1}}{\gamma} \tag{12}$$

It follows that the dynamics of the benchmark portfolio return *M* can be expressed as

$$\frac{dM}{M} = [r + (\mathbf{\Phi}_M)' (\boldsymbol{\mu} - r \cdot \mathbf{1})]dt + (\mathbf{\Phi}_M)' \boldsymbol{\sigma} d\mathbf{Z}. \tag{13}$$

if value function J is strictly concave with respect to gross return and the conditional expectation of the terminal utility of the portfolio's excess return over the benchmark, given the current information Ω_t. We denote the value function starting from time *t* to the end of the investment horizon *T* as

$$J(R,M,t) = \max E[U(R(T) - M(T)) \,|\, \Omega_t] \tag{14}$$

Assuming J is differentiable with respect to t and twice continuously differentiable with respect to *R* and *M*. In solving for the optimal portfolio strategy, we adopt the standard stochastic control approach. The principle of optimal stochastic control leads to the following Hamilton-Jacobi-Bellman (HJB) equation for the indirect utility function J:

$$J_t + \max_{\mathbf{\Phi}^{R-M}} \{(r + \mathbf{\Phi}'(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x))RJ_R + (r + \mathbf{\Phi}_M{}'(\boldsymbol{\mu} - r \cdot \mathbf{1}))MJ_M + \frac{1}{2}\mathbf{\Phi}'(\boldsymbol{\sigma}_x \boldsymbol{\sigma}_x')\mathbf{\Phi} R^2 J_{RR}$$

$$+ (\mathbf{\Phi})'(\boldsymbol{\sigma}_x \boldsymbol{\sigma}_x')(\mathbf{\Phi}_M) RMJ_{RM} + \frac{1}{2}(\mathbf{\Phi}_M)'(\boldsymbol{\sigma}_x \boldsymbol{\sigma}_x')(\mathbf{\Phi}_M) M^2 J_{MM}\} = 0 \tag{15}$$

with boundary condition $J(R,M,T) = U(R - M)$.

Now we adjust the portfolios by focusing on the tradeoff between excess return and variability relative to the benchmark. Since the risk of an adjusted portfolio is characterized by the performance of the selected bench mark, it may be more desirable to focus on the variation of the optimal portfolio over the benchmark. Roll (1992) discussed this approach under the standard mean variance framework and concluded that tracking error of the mean variance optimal portfolio is not necessarily mean-variance efficient.

4.2 Growth Optimal Portfolio

After deriving the optimal weights in terms of the indirect utility function which is also called the optimal value function, we can then explicitly obtain the optimal value in terms of the returns on the benchmark portfolio and the growth optimum. The optimal portfolio return shows that actively managed portfolio can be of value added over the benchmark portfolio on a risk adjusted basis. No arbitrage is often associated with the existence of a state price density or equivalent martingale measure, see Duffie (2001). In the benchmark framework, the object used as a reference unit for pricing purposes is a portfolio having a maximal growth rate, the growth optimal portfolio (GOP). Under the standard assumptions of risk-neutral pricing the inverse of the discounted GOP is the Radon-Nikodym derivative of a risk-neutral measure with respect to the empirical measure. Under the weaker assumptions made in the benchmark framework, in particular, the definition of arbitrage, such a risk-neutral measure may not exist, whereas it is still possible to define the GOP and apply it for pricing purposes as done in Platen (2002, 2004).

In the previous standard utility maximization with Constant Relative Risk Aversion, the optimal policies are all constant proportion portfolio allocation strategies. The portfolio is continuously rebalanced so as to always keep a constant proportion of the total fund value in the various asset classes, regardless of the level of the fund. Although such rebalance policies have a variety of optimality properties for the ordinary portfolio problem and are widely used in asset allocation practice, some investors deny to use constant proportion strategies in the belief that their individual intuition would suggest otherwise (see Grinold and Kahn (2000), Perold (1988)). Browne (1999) studied active portfolio management problems with objective related to the achievement of performance goals and shortfalls. Intuitively, a general incomplete market case manager of actively managed mutual funds is interested in shifting the investment policy with changes of returns on both their investment portfolios and the benchmark portfolio from time to time.

A well-known result about the growth optimum is that, in a long run, it can beat any portfolio with probability 1. This produces a valuable hint for investing in the growth optimum portfolio over a long term investment horizon. In the setting of Merton (1971) the vector of the growth optimum portfolio weights is

$$\mathbf{\Phi}_G = (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x), \tag{16}$$

where $\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x$ is the instantaneous risk premium with constant event jump arrival intensity λ and $(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}$ is an invertible matrix for the optimal portfolio.

The vector $\mathbf{\Phi}_G$ plays a fundamental role in the theory of finance (see Merton 1992, Ch. 6) and will also play a fundamental role in the sequel. Following Merton (1990), we refer to the vector $\mathbf{\Phi}_G$ as the optimal-growth portfolio strategy. The reason for this is the proportion of the investor's wealth invested in the risky stock at time *t*, has many optimality properties associated with it in an ordinary portfolio setting (there is no benchmark) that are relevant to growth-related objectives. In particular, for an investor whose wealth evolves according to the wealth process in

Equation (9), and who is not concerned with performance relative to any benchmark, (i) $\mathbf{\Phi}_G$ maximizes the expected logarithm of terminal wealth, for any fixed terminal time *T*, hence (ii) $\mathbf{\Phi}_G$ maximizes the actual and expected rate at which the wealth compounds. Moreover, perhaps, and certainly more relevant to our concerns here is the property (iii) $\mathbf{\Phi}_G$ minimizes the expected time until any given level of wealth is achieved. The policy $\mathbf{\Phi}_G$ has extended optimality properties in our benchmark-based model that minimizes the expected time until the benchmark portfolio strategy is beaten by any given percentage. The fact that it does not provide any insight for the active portfolio manger as to the role the benchmark plays in the investment decision, and as such might not be a reasonable objective for an active portfolio manager. To eliminate redundant assets, the dynamics of the growth return *G* of the optimal portfolio at time *t* is

$$\frac{dG}{G} = [r + (\mathbf{\Phi}_G)'(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x)]dt + (\mathbf{\Phi}_G)' \boldsymbol{\sigma} d\mathbf{Z}. \tag{17}$$

We take a brief review of the growth on optimal portfolio that maximizing expected log utility of wealth is equivalent to maximizing the geometric growth rate. The optimal expected growth rate, $E[\frac{1}{T}\ln G_{(t)}]$, defined as the average of the optimal value function over the investment horizon, is equal to

$$E[\frac{1}{T}\int_0^T (r + \frac{1}{2}(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x)'(\boldsymbol{\sigma\sigma}')^{-1}(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x))dt] \tag{18}$$

With $(\boldsymbol{\sigma\sigma}')^{-1}$ and $(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x)$ being constant, the optimal growth rate is

$$r + \frac{1}{2}(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x)' (\boldsymbol{\sigma\sigma}')^{-1}(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x). \tag{19}$$

4.3 Solution for Optimal Portfolio Return

The active portfolio management problems considered for an incomplete market in Browne (1999) are: (i) maximizing the probability that the performance goal is reached before shortfall occurs; (ii) minimizing the expected time until the performance goal is reached; (iii) maximizing the expected time until shortfall is reached; (iv) maximizing the expected discounted reward obtained upon achieving the goal; and (v) minimizing the expected discounted penalty paid upon falling to shortfall level. Since the utility function is strictly concave, the optimal value function *F* is also strictly concave in *R*. In solving for the optimal portfolio strategy, we adopt the standard stochastic control approach. Since the utility function *U* is strictly concave, the optimal value function J is also strictly concave in R. The first order condition implies that the optimal solution, $\mathbf{\Phi}^*$, to the embedded maximization problem in equation (16),

$$\begin{aligned}\sup_{\mathbf{\Phi}}\{&(r + \mathbf{\Phi}'(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x))RJ_R + (r + \mathbf{\Phi}_M{}'(\boldsymbol{\mu} - r \cdot \mathbf{1}))MJ_M + \frac{1}{2}\mathbf{\Phi}'(\boldsymbol{\sigma}_x\boldsymbol{\sigma}_x')\mathbf{\Phi}R^2 J_{RR} \\ &+ (\mathbf{\Phi})'(\boldsymbol{\sigma}_x\boldsymbol{\sigma}_x')(\mathbf{\Phi}_M)RMJ_{RM} + \frac{1}{2}(\mathbf{\Phi}_M)'(\boldsymbol{\sigma}_x\boldsymbol{\sigma}_x')(\mathbf{\Phi}_M)M^2 J_{MM}\}.\end{aligned} \tag{20}$$

which must satisfy the following equation

$$(\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x))RJ_R + (\boldsymbol{\sigma}_x\boldsymbol{\sigma}_x')\boldsymbol{\Phi}^* R^2 J_{RR} + (\boldsymbol{\sigma}_x\boldsymbol{\sigma}_x')\boldsymbol{\Phi}_M RMJ_{RM} = 0 \qquad (21)$$

For $\boldsymbol{\Phi}_G = (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x)$, the optimal solution is

$$\boldsymbol{\Phi}^* = -\frac{J_R}{RJ_{RR}}\boldsymbol{\Phi}_G - \frac{M}{R}\frac{J_{RM}}{J_{RR}}\boldsymbol{\Phi}_M . \qquad (22)$$

The optimal portfolio can be decomposed into two parts. The first part is the growth optimum, and the second part is a hedging component of the stochastic investment opportunities. We have adopted an incomplete market asset pricing model as in He and Pearson (1991). Now, we substitute the optimal solution $\boldsymbol{\Phi}^*$ into the optimality equation (15) to yield the following partial differential equation

$$J_t + rRJ_R + rMJ_M + \beta MJ_M - \frac{\beta MJ_R J_{RM}}{J_{RR}} + \frac{1}{2}\left(-\frac{\alpha J_R^2}{J_{RR}} - \frac{\theta M^2 J_{RM}}{J_{RR}} + \theta M^2 J_{MM}\right) = 0 \qquad (23)$$

where $\alpha = (\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x)'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x)$ which can be interpreted as the market price for aggregated risk, $\beta = \boldsymbol{\Phi}_M'(\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x)$ and $\theta = \boldsymbol{\Phi}_M'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\boldsymbol{\Phi}_M$. Let $K(R,\ M,\ t\ ) = (R - M)\cdot e^{-\int_t^T rds}$. By conjecturing the indirect utility function is of the form

$$J(R,\ M,\ t\ ) = U(K)f(t) + g(t), \qquad (24)$$

where $f(t)$ and $g(t)$ satisfy the necessary boundary conditions $f(T)$=1 and $g(T)$=0, respectively. Given this functional form, we take derivatives of $J(R,\ M,\ t\ )$ with respect to its arguments, $J_R = \frac{dU(K)}{U(K)} f(t)\cdot e^{\int_t^T rds}$, and $J_{RR} = -J_{RM} = \frac{d^2U(K)}{U^2(K)} f(t)\cdot e^{2\int_t^T rds}$, where $\frac{dU(K)}{U(K)}$ and $\frac{d^2U(K)}{U^2(K)}$ stand for the first order and the second order mathematical derivatives, respectively. Substituting the previous partial derivatives into the HJB equation in Equation (23) and using a specific utility function of the HARA type

$$-\frac{dU(\phi)}{U(\phi)} / \frac{d^2U(\phi)}{U^2(\phi)} = \phi + c > 0, \qquad (25)$$

where $\frac{dU(\phi)}{U(\phi)}$ and $\frac{d^2U(\phi)}{U^2(\phi)}$ stand for the first order and the second order mathematical derivatives of portfolio weight ϕ, respectively.

in this case, we have $U(\phi) = a\ln(\phi + c) + b$ and the first order $\frac{dU(\phi)}{U(\phi)} = \frac{a}{\phi + c}$, where $a \neq 0$ is a conforming constant such that the function $U(\phi)$ is strictly concave within the domain and *b* is an arbitrary constant. We can reduce the optimality equation in (24) as

$$U(K)\frac{df(t)}{dt} + \frac{dg(t)}{dt} + \frac{1}{2}\alpha(K + c)\frac{dU(K)}{U(K)} f(t) = 0 \qquad (26)$$

where K is the function $(R - M)\cdot e^{-\int_t^T rds}$ and the value α is actually equal to the risk premium of the growth optimum in the risky assets. If the market portfolio in the risky asset is equivalent to the growth optimal portfolio, α should be viewed as the risk premium for the overall market. The solution in accordance with utility function form, by substituting, Equation (26) becomes

$$(a\ln(K+c)+b)\frac{df(t)}{dt}+\frac{dg(t)}{dt}+\frac{1}{2}\alpha\cdot a\cdot f(t)=0\,. \tag{27}$$

Similarly, it must be true that $\frac{df(t)}{dt}=0$ and $\frac{dg(t)}{dt}=-\frac{1}{2}\alpha\cdot a\cdot f(t)$, with the boundary condition $f(T)=1$ and $g(T)=0$, it follows that

$f(t)=1$ and $g(T)=-\frac{1}{2}a\int_t^T \alpha ds$. Therefore, substituting the obtained $f(t)$ and $g(t)$ back into equation (24) yields the optimal value function

$$J(R,M,t)=a(\ln(e^{\int_t^T rds}(R-M)+c)-\frac{1}{2}\int_t^T \alpha ds)+b \tag{28}$$

The parameters, *a*, *b* and *c* are constant. As for the embedded maximization, it is required that the optimal value function *J* be strictly concave in *R*.

Having obtained the optimal value function, we now explicitly derive the optimal portfolio weight at each point in time. As a result of direct calculation using Equation (22) and (27), for any utility function of the HARA type as defined in Equation (25), the corresponding optimal weights are

$$\mathbf{\Phi}^*_{active}=\frac{R-M+ce^{-\int_t^T rds}}{R}\mathbf{\Phi}_G+\frac{M}{R}\mathbf{\Phi}_M\,. \tag{29}$$

Similar to Browne (2000), our actively managed portfolio is also a dynamic linear combination of weights in the growth optimum and the benchmark. However, the percentages allocated to the growth optimal portfolio and the benchmark must depend independently on the two state variables contrary to Browne (2000) in which the allocation depends on the single state variable, the ratio of the actively managed portfolio return and the benchmark portfolio return. When the utility function is logarithmic (*c*=0), the two optimal investment policies are identical. In this special case, by minimizing the time to achieve some target return before dropping to a subsistent level is equivalent to maximizing the expected logarithmic of return. The investors will solely split the funds in the benchmark and the growth optimum.

However, the optimal weights are dynamically changing and depending on the actual outcome of the returns on the benchmark and the growth optimum. Our model is more general than Browne (2000) in a sense that it provides the investment managers/investors with more flexible allocation strategies depending on the two state variables independently. Since *M*(0)=*R*(0)=1, (if $c\neq 0$), the managers/investors will initially invest all funds in the benchmark and finance the growth optimum portfolio through the cash bond (riskfree asset) if required.

The optimal weight in the benchmark portfolio dynamically depends on the ratio, $\frac{M}{R}$, of the performance of the benchmark to that of the actively managed portfolio. It is true that if the benchmark performs better than the fund, the fund then allocates more in the benchmark portfolio than in the growth optimum. With $e^{-\int_t^T rds}$ being interpreted as the price of the cash bond, the optimal allocation to the growth optimum depends not only on the ratio $\frac{M}{R}$, but also on the ratio $\frac{e^{-\int_t^T rds}}{R}$ of the returns on the actively managed fund. The sum of the weights in the benchmark portfolio and the growth optimum is $\frac{R-M+ce^{-\int_t^T rds}}{R}+\frac{M}{R}=1+\frac{ce^{-\int_t^T rds}}{R}$ which is greater or less than one depending on the sign of the constant *c*.

4.4 Criteria for Adjusting Portfolio Choice

We now turn to the discussion of the optimal portfolio return over time. Since the optimal weight vector is a linear combination of the growth optimum and the benchmark portfolio as in Equation (29), one might imagine that the optimal portfolio return should be expressible in terms of the returns on the growth optimum and the benchmark portfolio. We find the relationship between the optimal portfolio return, growth optimum returns and benchmark portfolio return by substituting the optimal weight from Equation (29) in (9), which yields

$$\frac{dR}{R}=[r+(\frac{R-M+c\cdot e^{-\int_t^T rds}}{R}\mathbf{\Phi}_G+\frac{M}{G}\mathbf{\Phi}_M)'(\boldsymbol{\mu}-r\cdot\mathbf{1})]dt$$

$$+(\frac{R-M+c\cdot\int_t^T rds}{R}\mathbf{\Phi}_G+\frac{M}{G}\mathbf{\Phi}_M)'(\boldsymbol{\mu}-r\cdot\mathbf{1})\boldsymbol{\sigma}d\mathbf{Z} \tag{30}$$

Combining with the dynamics of the growth optimum portfolio (17), it follows that

$$d(R-M)=[r(R-M)+(R-M+c\cdot e^{-\int_0^t rds})(\mathbf{\Phi}_M)'(\boldsymbol{\mu}-r\cdot\mathbf{1})dt$$

$$+(R-M+c\cdot e^{-\int_0^t rds})(\mathbf{\Phi}_M)'(\boldsymbol{\mu}-r\cdot\mathbf{1})\boldsymbol{\sigma}d\mathbf{Z} \tag{31}$$

The direct application of the Ito's formula for a given stochastic process $\frac{dP_t}{P_t}=\mu_t dt+\sigma_t dZ_t$, at the most time varying with $q_0\neq 0$, the solution to the stochastic differential equation $\frac{dP_t}{P_t}=k\frac{dQ_t}{Q_t}$ is given as

$$P_t=\frac{P_0}{Q_0}Q_t^k e^{\frac{1}{2}k(1-k)\int_0^t \sigma_s\sigma_s' ds}, \tag{32}$$

for any constant *k*. According to the dynamics of the growth optimum portfolio return that implies

$$\frac{d(G\cdot e^{-\int_0^t rds})}{(G\cdot e^{-\int_0^t rds})}=\mathbf{\Phi}_G{}'(\boldsymbol{\mu}-r\cdot\mathbf{1}-\lambda\boldsymbol{\mu}_x)dt+\mathbf{\Phi}_G{}'\boldsymbol{\sigma}d\mathbf{Z} \tag{33}$$

Therefore, by (32) if $k\neq 0$,

$$\frac{d((r-M)e^{-\int_0^t rds}+\frac{c}{k}\cdot e^{-\int_0^T rds})}{(r-M)e^{-\int_0^t rds}+\frac{c}{k}\cdot e^{-\int_0^T rds}}=k\cdot\frac{d(G\cdot e^{-\int_0^t rds})}{(G\cdot e^{-\int_0^t rds})} \tag{34a}$$

then

$$(R-M)e^{-\int_0^t rds}+\frac{c}{k}\cdot e^{-\int_0^T rds}=\frac{c}{k}\cdot e^{-\int_0^T rds}(G\cdot e^{-\int_0^t rds})^k e^{\frac{1}{2}k(1-k)\int_0^t \alpha ds} \tag{34b}$$

Thus, the relationship is expressed by

$$R=M+\frac{c}{k}\cdot e^{-\int_t^T rds}(G^k\cdot e^{-k\int_0^t rds+\frac{1}{2}k(1-k)\int_0^t \alpha ds}-1) \tag{35}$$

where *G* is the gross return of the growth optimum portfolio with $G(0)=1$.

Owing to the optimal portfolio return is closely related to the return of the benchmark portfolio. The excess return of the actively managed portfolio is paid off on a risk adjusted basis. The added value is determined by the risk aversion level and the investment horizon. Thus, the added values are related to the risk aversion level and the investment horizon under the reward and variability are essential component in investment. If the investor's utility function $U(\phi)$ is of constant relative risk aversion, the return of an actively managed portfolio is identical to the return on the benchmark. In other words, the passive portfolio management strategy is optimal for constant relative risk aversion (CRRA) investors. Thereafter active portfolio managements for CRRA utilities, such as standard logarithmic and power utilities, and the exponential utilities do not add any value over the benchmark performance, since the optimal portfolio for these investors is identical to the benchmark. However, some of the utility functions of the hyperbolic absolute risk aversion (HARA) type do generate different optimal portfolios which are different from the benchmark, even though popular investment approach of mean variance analysis which explicitly characterizes the tradeoff between return and risk falls into this category of utility function. With risk aversion index γ, let ϕ_γ and R_γ denote the optimal weight vector and the optimal gross return, respectively. By Equation (29) the optimal weight vector is

$$\mathbf{\Phi}_\gamma^*=\frac{M-R_\gamma+\frac{1}{\gamma}e^{-\int_t^T rds}}{R_\gamma}\mathbf{\Phi}_G+\frac{M}{R_\gamma}\mathbf{\Phi}_M\,. \tag{36}$$

We find the criteria for adjusting portfolio choice focus on the tradeoff between excess return and variability relative to a benchmark. Since the risk of an actively managed portfolio is characterized by the performance of the selected benchmark, it may be more desirable to focus on the variation of the optimal portfolio over the benchmark. For instance, if the excess return of a portfolio is constant and equal to a negative value, the risk in the direction of error mean variance analysis would be quantified as zero, which is certainly unacceptable. To deviate from the variance measure, we use the variability which is defined as the mean square error between the actively management portfolio and the benchmark. For risk aversion level, γ, the objective function is defined as

$$\max\ E[R-M]-\frac{1}{2}\gamma E[R-M]^2 \tag{37}$$

which is equivalent to maximizing the expected quadratic utility of the excess return in the form $U(\phi)=\phi-\frac{1}{2}\gamma\phi^2$.

From the general form of the HARA utility function (27), the corresponding parameters, k, a, b and c depend on the risk aversion index, γ and are equal to -1, $-\gamma$, $\frac{1}{2\gamma}$ and $\frac{1}{\gamma}$, respectively. The optimal gross portfolio return depends on the risk aversion index is the benchmark plus some value of excess return related to the growth optimum on a risk adjusted basis. Explicitly by Equation (35), the optimal gross return with risk aversion index γ, is

$$R_\gamma = M + \frac{1}{\gamma} e^{-\int_t^T r ds} (1 - G^{-1} e^{\int_0^t r ds - \int_0^t \alpha ds}) \tag{38}$$

with $\frac{1}{\gamma}$ being interpreted as risk tolerance. Equation (38) implies that the excess return or value added is proportional to the risk tolerance, with a ratio being negatively sloped linear function of the inverse of growth optimum return. It implicates the higher growth optimum return, the greater the marginal excess return per risk tolerance. With growth return $G > e^{-\alpha T}$, a positive premium is added by using adjusted portfolio management. Otherwise, $G < e^{-\alpha T}$, a negative premium is added. When the event jump incurs the risk premium $(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x)$ varying, thus $\alpha = (\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x)' (\boldsymbol{\sigma\sigma}')^{-1} (\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x)$ forms a critical value in the optimal gross return when there are event jumps. Whether the adjusted portfolio achieves a positive excess return is dependent on the growth optimal portfolio, which can be viewed as a reference portfolio. Henceforth if the event jump $\lambda \boldsymbol{\mu}_x$ to downward side α will compensate the jump risk, otherwise the event jump $\lambda \boldsymbol{\mu}_x$ to upward side α will reduce the compensation of the jump risk. When the risk premium on event jump portfolio $(\boldsymbol{\mu} - r \cdot \mathbf{1} - \lambda \boldsymbol{\mu}_x)$ is less than the optimal estimates about the excess return ($\mathbf{m} \; - r \cdot \mathbf{1}$), the investment will vary with the risk premium.

5. Numeric Example

In practice, many individual professional and institutional investors follow the benchmarking procedure: for example, many mutual funds take the Standard and Poors (S&P) 500 Index as a benchmark; commodity funds seek to beat the Goldman Sachs Commodity Index; bond funds try to beat the Lehman Brothers Bond Index/ etc. Moreover, benchmarking is not specific to professional investors, as many individual investors have implicitly followed a benchmarking procedure, for example, by trying to beat inflation, exchange rates, or other indices. In other applications, such as pension funds, the benchmark might be a liability, see Litterman and Winkelman (1996). For a treatment of active portfolio management in a static setting, see Grinold and Kahn (2000).

We now calibrate a world stock portfolio that is a diversified portfolio to approximate the implementation. The paper employs monthly data over the period from January, 1997 to April, 2005 for a sample of country event risk on stock returns of BRICs. For comparative purposes, a sample of stock returns on developed markets of US, UK and Japan is also considered. We investigate the variety of stock returns without considering dividend yields. Since the literature can be used generally to aid in the identification of local risk factors that might play important roles across emerging markets. For instance, Bekaert and Harvey (2000) find dividend yields to be important determinants of local equity market variation in emerging markets, and further argue that dividend yields may proxy for capital market liberalization. Similar associations between returns and dividend yields have been reported for developed markets.

Dividend yields forecast both future dividend changes and future stock returns when the expected returns vary through time. They are often used because permanent changes in costs of capital are more likely to be reflected in dividend yields than in returns. Further, dividend yields exhibit less variability than stock returns and hence are more likely to represent permanent, rather than transitory changes (Bekaert and Harvey, 2000). For this reason, we study the instantaneous movement in mean and standard deviation of stock return excluding the dividend yield.

Stock values fluctuate in response to the activities of individual companies, and general market and economic conditions. One may have a gain or loss when one sell his shares. In addition to those general risks, international investing involves different or increased risks. The performance of international funds depends upon currency values, political and regulatory environments, and overall market and economic factors in the countries in which they invest. The risks are magnified in countries with emerging markets, since these countries may have relatively unstable governments and less established markets and economies. Suppose the investment opportunities consist of a riskless asset earning r=2% per annum and risky assets of seven countries, the data for US, UK, Japan, Brazil, Russia, India and China are retrieved from NASDAQ composition, London-FTSE-100 Index, Tokyo Nikkei 225 Index, Brazil Bovesp Index, Moscow Times Stock Index, Bombay Stock Exchange Index, and Shanghai Synthesis Index respectively. By investigating the monthly data period under the condition of political instability firstly, we present the parameter estimates for a jump-diffusion model of BRICs as specified by Press (1967) and summarize the statistical properties of the stock price return for US, UK and Japan in Table 1.

Table 1. Statistical Parameter Estimates for Stock Price Return

Region	μ	σ^2	$\hat{\mu}_x$	$\hat{\sigma}_x^2$	$\hat{\lambda}$
Developing countries:	0.0132	0.0120	0.0311	0.0116	0.4233
Brazil			(-0.2289)		(0.0575)
Russia	0.0348	0.0157	0.1621	0.0135	0.2147
			(-0.3708)		(0.0939)
India	0.0066	0.0068	0.1878	0.0041	0.2632
			(-0.0251)		(0.0352)
China	0.0016	0.0045	0.4347	0.0131	0.0025
			(-0.6416)		(0.0036)
Developed countries:					
US	0.0066	0.0083	--	--	--
UK	0.0007	0.0019	--	--	--
Japan	-0.005	0.0032	--	--	--

Note: μ is the mean of stock return, σ^2 is the instantaneous variance of stock return, and a Poisson process with stochastic estimated arrival intensity $\hat{\lambda}$. A random percentage jump size with estimated mean return $\hat{\mu}_x$, and $\hat{\sigma}_x^2$ is the estimated variance of random percentage jump size of stock price return.

S_{Brazil}, S_{Russia}, S_{India}, S_{China}, S_{US}, S_{UK} and S_{Japan} represent the stock price dynamics of Brazil Bovesp Index, Moscow Times Stock Index, Bombay Stock Exchange Index, Shanghai Synthesis Index, NASDAQ composition, London-FTSE-100 Index, and Tokyo Nikkei 225 Index, respectively with the following process:

$$\begin{bmatrix} \frac{dS_{Brazil}}{S_{Brazil}} & \frac{dS_{Russia}}{S_{Russia}} & \frac{dS_{India}}{S_{India}} & \frac{dS_{China}}{S_{China}} & \frac{dS_{US}}{S_{US}} & \frac{dS_{UK}}{S_{UK}} & \frac{dS_{Japan}}{S_{Japan}} \end{bmatrix}' =$$

$$\begin{bmatrix} 0.0132 & 0.0120 & 0.0077 & 0.0038 & 0.0006 & 0.0061 & 0.0029 & 0.0024 \\ 0.0348 & 0.0077 & 0.0157 & 0.0023 & 0.0017 & 0.0045 & 0.0025 & 0.0024 \\ 0.0066 & 0.0038 & 0.0023 & 0.0068 & 0.0009 & 0.0029 & 0.0007 & 0.0017 \\ 0.0016 & 0.0006 & 0.0017 & 0.0009 & 0.0045 & 0.0004 & -0.0002 & 0.0005 \\ 0.0067 & 0.0062 & 0.0045 & 0.0029 & 0.0004 & 0.0083 & 0.0025 & 0.0027 \\ 0.0007 & 0.0029 & 0.0025 & 0.0007 & -0.0002 & 0.0025 & 0.0019 & 0.0010 \\ -0.005 & 0.0024 & 0.0024 & 0.0016 & 0.0005 & 0.0027 & 0.0010 & 0.0032 \end{bmatrix} \cdot \begin{bmatrix} dt \\ dZ_1 \\ dZ_2 \\ dZ_3 \\ dZ_4 \\ dZ_5 \\ dZ_6 \\ dZ_7 \end{bmatrix} \quad (39)$$

$$(\boldsymbol{\sigma\sigma}') = \begin{bmatrix} 0.012 & 0.0137 & 0.0090 & 0.0073 & 0.0099 & 0.0047 & 0.0046 \\ 0.0137 & 0.0157 & 0.0103 & 0.0084 & 0.0114 & 0.0054 & 0.0071 \\ 0.0090 & 0.0103 & 0.0068 & 0.0055 & 0.0075 & 0.0036 & 0.0046 \\ 0.0073 & 0.0084 & 0.0055 & 0.0045 & 0.0061 & 0.0029 & 0.0038 \\ 0.0099 & 0.0114 & 0.0075 & 0.0061 & 0.0083 & 0.0039 & 0.0051 \\ 0.0047 & 0.0055 & 0.0036 & 0.0029 & 0.0039 & 0.0019 & 0.0024 \\ 0.0046 & 0.0071 & 0.0046 & 0.0038 & 0.0051 & 0.0025 & 0.0032 \end{bmatrix} \quad (40)$$

(40) implies that the inverse matrix of covariance matrix is

$$(\boldsymbol{\sigma\sigma}')^{-1} = 1.0\text{e}{+}004 * \begin{bmatrix} -0.0073 & 0.0052 & -0.0055 & 0.1359 & -0.0324 & -0.0073 & -0.0931 \\ 0.0251 & 0.1689 & -0.1499 & -0.2627 & 0.6094 & -1.2007 & 0.0655 \\ 0.1030 & -0.1640 & 0.3199 & -2.0101 & 0.3175 & 1.4839 & 0.4282 \\ -0.0803 & -0.1161 & -0.0425 & 1.7275 & -0.3361 & -1.0014 & -0.2612 \\ -0.0074 & 0.0151 & 0.2809 & 0.1639 & -0.4146 & 0.0983 & -0.0305 \\ -0.0452 & -0.0789 & -0.7411 & 1.6457 & -0.9869 & 1.7299 & -0.3569 \\ -0.0512 & 0.0290 & 0.0522 & -0.3024 & 0.0604 & 0.2366 & 0.0378 \end{bmatrix} . \quad (41)$$

For simplicity without loss of the generality, given $\gamma = 3$, we calculate the benchmark portfolio weights with no jumps in the risky assets by

$$\mathbf{\Phi}_M^* = (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\frac{\boldsymbol{\mu} - r\cdot\mathbf{1}}{\gamma} = \frac{1}{3}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\begin{bmatrix} -0.0068 \\ 0.0148 \\ -0.0134 \\ -0.0184 \\ -0.0134 \\ -0.0193 \\ -0.025 \end{bmatrix} = \begin{bmatrix} 2.0031 \\ 75.1365 \\ -46.7615 \\ -6.7523 \\ -6.9445 \\ -108.1713 \\ -2.2607 \end{bmatrix} \tag{42}$$

Thus, the dynamics of the benchmark M portfolio return is,

$$\frac{dM(t)}{M(t)} = [r + (\mathbf{\Phi}_M)'(\boldsymbol{\mu} - r\cdot\mathbf{1})]dt + (\mathbf{\Phi}_M)'\boldsymbol{\sigma}d\mathbf{Z} = 4.1065\,dt +$$

$$\begin{bmatrix} 0.2193 & 9.4143 & -3.8561 & -0.4529 & -0.6326 & -4.7141 & -0.1278 \end{bmatrix}\begin{bmatrix} dZ_1 \\ dZ_2 \\ dZ_3 \\ dZ_4 \\ dZ_5 \\ dZ_6 \\ dZ_7 \end{bmatrix} \tag{43}$$

For the risk premium, $[\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x] = \begin{bmatrix} -0.0199,-0.02,-0.0628,-0.0195,-0.0134,-0.0193,-0.025 \end{bmatrix}^t$ with upward jumps, and

$[\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x] = \begin{bmatrix} 0.0063,0.0496,-0.0125,-0.0161,-0.0134,-0.0193,-0.025 \end{bmatrix}^t$ with downward jumps, and r =0.02.

By Equation (16)

$$\mathbf{\Phi}_G = (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}[\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x] = \begin{bmatrix} -20.1934, 135.1222, 212.9423, -286.8272, -235.0969, 134.5251, 34.2799 \end{bmatrix}^t$$

with upward jumps,

and $\begin{bmatrix} -16.5857,174.9614,218.4717,-351.4773,-79.6417,-249.2130,57.0151 \end{bmatrix}^t$ with downward jumps.

$\alpha = [\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x]'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(\boldsymbol{\mu} - r\cdot\mathbf{1} - \lambda\boldsymbol{\mu}_x)$ =-11.7172 with upward jumps, or 10.5106 with downward jumps.

By Equation (17)

$$\frac{dG(t)}{G(t)} = [r + \alpha]dt + (\mathbf{\Phi}_G)'\boldsymbol{\sigma}d\mathbf{Z}$$

$$= -11.6972\,dt + [-2.211 \;\; 16.930 \;\; 17.559 \;\; -19.241 \;\; -21.418 \;\; 5.862 \;\; 1.939]d\mathbf{Z} \tag{44}$$

with upward jumps, or $10.5306\,dt + [-0.242 \;\; 2.1213 \;\; 1.4480 \;\; -1.2906 \;\; -1.9513 \;\; 0.2555 \;\; 0.1096]d\mathbf{Z}$ with downward jumps. We then pass the function needed to solve the first-order system of Ordinary Differential Equation.

We obtain $M = e^{4.1065t-0.1511Z}$ and $G = e^{-11.6972t-0.57774Z}$ with upward jumps, or $G = e^{10.5306t+0.45053Z}$ with downward jumps, assuming the investment horizon is a year (i.e. T=1). Note that the optimal gross portfolio return, dependent on the risk aversion index, is the benchmark plus some value excess return related to the growth optimum on a risk adjusted basis. Explicitly by Equation (36), the optimal portfolio weights in the risky assets are the vector $\mathbf{\Phi}_\gamma^* = \dfrac{M - R_\gamma + \frac{1}{\gamma} e^{-\int_t^T r ds}}{R_\gamma} \mathbf{\Phi}_G + \dfrac{M}{R_\gamma} \mathbf{\Phi}_M$

$$= 6.0596 \times 10^{-5} \begin{bmatrix} -20.1934 \\ 135.1222 \\ 212.9423 \\ -286.8272 \\ -235.0969 \\ 134.5251 \\ 34.2799 \end{bmatrix} + 0.994656 \begin{bmatrix} 2.0031 \\ 75.1365 \\ -46.7615 \\ -6.7523 \\ -6.9445 \\ -108.1713 \\ -2.2607 \end{bmatrix} \tag{45}$$

with upward jumps, or $6.0596 \times 10^{-5} \begin{bmatrix} -16.5857 \\ 174.9614 \\ 218.4717 \\ -351.4773 \\ -79.6417 \\ -249.2130 \\ 57.0151 \end{bmatrix} + 0.994656 \begin{bmatrix} 2.0031 \\ 75.1365 \\ -46.7615 \\ -6.7523 \\ -6.9445 \\ -108.1713 \\ -2.2607 \end{bmatrix}$ with downward jumps.

Sensitivity analysis of optimal gross return R_γ, by simulating the equation (38) is given in Table 2.

Table 2. Risk Adjusted Basis Sensitivity Analysis of Optimal Gross Return

(Jump Once Only λ =1)

Risk Aversion Parameter	Jump size	Optimal Gross Return R_γ		
		$t = \frac{1}{10}$	$t = \frac{1}{2}$	$t = \frac{9}{10}$
0.5	0.9	6.4661	5.9492	5.5722
	0.2	14.4915	14.5153	14.5391
	0	2.4719	2.4719	2.4719
	-0.2	-5.1247	-5.1012	-5.0776
	-0.9	2.4324	2.9847	3.3934
3.0	0.9	1.0944	1.0082	0.9454
	0.2	2.4319	2.4359	2.4399
	0	0.4287	0.4287	0.4287

	-0.2	-0.8375	-0.8335	-0.8296
	-0.9	0.4221	0.5141	0.5822

Table 2 shows that 20% downward jumps incur the negative gross return and risk aversion deserves preserved gross return. The gross return grows with time when jumps occur, except for the 90% jump size.

6. Conclusion

In the search for the appropriate criteria to be used for adjusting portfolio weights, we reveal some interesting findings. First, the critical criterion for adjusting the actively diversified portfolio is the survival rate of portfolio return below which the active portfolio under performs the selected benchmark. The optimal weight in the benchmark portfolio is dynamically dependent on the ratio of the performance of the benchmark to that of the actively managed portfolio. We also discover that if the benchmark performs better than the actively managed fund, the fund manager allocates more in the benchmark than in the growth optimum.

With the survival rate, a compound rate being interpreted as the price of the bond, the optimal allocation to the growth optimum depends not only on the ratio of the benchmark to that of the actively managed portfolio, but also on the ratio of the returns on the money market fund and the actively managed fund.

Whether the sum of the weights in the benchmark portfolio and the growth optimum is greater or less than 1 is dependent on the sign of the constant term in equation (38).

Appendix

Using the relationship between cumulants and moments (Kendall and Stuart, 1963), it can be proven that this distribution is leptokurtic and therefore might better describe the actual stock price return behavior than the pure lognormal model.

Let k_i denotes the i-th cumulant, the kurtosis for a Poisson mixture of normal distribution is written as $\frac{k_4}{k_2^2} = \frac{\lambda\tau\left(3\sigma_x^4 + 6\mu_x^2\sigma_x^2 + \mu_x^4\right)}{\left(\sigma^2\tau + \lambda\tau\sigma_x^2 + \lambda\tau\mu_x^2\right)^2}$. It is easily seen to be positive as $\frac{k_4}{k_2^2}$ is zero for the normal distribution. The skewness has the sign of μ_x, $\frac{k_3}{k_2^{3/2}} = \frac{\lambda\tau\mu_x\left(3\sigma_x^2 + \mu_x^2\right)}{\left(\sigma^2\tau + \lambda\tau\sigma_x^2 + \lambda\tau\mu_x^2\right)^{3/2}}$, $\frac{k_3}{k_2^{3/2}}$ =0, if μ_x =0. Press (1967) assumes that the diffusion component has zero drift (μ =0), thereby reducing the number parameters to be estimated to four（$\hat{\lambda}$ ，$\hat{\sigma}^2$ ，$\hat{\sigma}_x^2$ ，$\hat{\mu}_x$）in the model. The parameter estimation approach employs the cumulant matching method which relies upon the theoretical relationship between the population's cumulants and the parameters of the data distribution. These four unknown variables can be derived by letting the relationship between the sample cumulants and the sample moments centered at around zero. It yields the solution by solving the following simultaneous equations:

$$
\begin{cases}
\mu_x^4 - 2\dfrac{\overline{k_3}}{\overline{k_1}}\mu_x^2 + \dfrac{3\overline{k_4}}{2\overline{k_1}}\mu_x - \dfrac{\overline{k_3}^2}{2\overline{k_1}^2} = 0 \\
\hat{\lambda} = \dfrac{\overline{k_1}}{\mu_x} \\
\hat{\sigma}_x^2 = \dfrac{\overline{k_3} - \mu_x^2\overline{k_1}}{3\overline{k_1}} \\
\hat{\sigma}^2 = \overline{k_2} - \dfrac{\overline{k_1}}{\mu_x}\left(\mu_x^2 + \dfrac{\overline{k_3} - \mu_x^2\overline{k_1}}{3\overline{k_1}}\right),
\end{cases}
$$

where $\overline{k_i}$ denotes the sample cumulant. The polynomial in μ_x has four roots: two complex and two real. Of these roots, the real root which yields a positive $\hat{\lambda}$ is chosen. Beckers (1981) considers that to obtain sensible parameter values some different prior restrictions must be applied to the diffusion mean. Therefore, the mean jump equal to zero is applied in estimation. This allows that at random intervals the stock price returns experience positive or negative jumps but presumes that the jumps average to zero. For the reasonable result in the impacts of jumps setting up the model, the mean return to the risky asset μ is independent of the jump occurrence. Thus, we apply the way that Press obtains his parameter estimates.

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