ANALYTIC SOLUTION FOR THE NUCLEOLUS OF A THREE-PLAYER COOPERATIVE GAME

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ABSTRACT

The nucleolus solution for cooperative games in characteristic function form is usually computed numerically by solving a sequence of linear programming (LP) problems, or by solving a single, but very large-scale, LP problem. This paper proposes an algebraic method to compute the nucleolus solution analytically (i.e., in closed-form) for a three-player cooperative game in characteristic function form. We first consider cooperative games with empty core and derive a formula to compute the nucleolus solution. Next, we examine cooperative games with non-empty core and calculate the nucleolus solution analytically for five possible cases arising from the relationship among different coalition values.

Key words: Three-player cooperative game in characteristic function form, nucleolus, linear programming.

EXTENDED ABSTRACT

Cooperative game theory studies situations involving multiple players who can cooperate and take joint actions in a coalition to increase their "wealth." The important problem of allocating the newly accrued wealth among the cooperating players in a fair manner has occupied game theorists since the 1940s. More than a dozen alternate solution concepts have been proposed to determine the allocation but only a few of these concepts have received the most attention. Von Neumann and Morgenstern [21] who were the originators of multiperson cooperative games proposed the first solution concept for such games known as the *stable set*. However, due to the theoretical and practical difficulties associated with it, the stable set concept fell out of favour. In 1953, Gillies [6] introduced the concept of *core* as the set of all undominated payoffs (i.e., imputations) to the players satisfying rationality properties. Even though the core has been found useful in studying economic markets, it does not provide a unique solution to the allocation problem. Also in 1953, Shapley [18] wrote three axioms which would capture the idea of a fair allocation of payoffs and developed a simple, analytic, expression to calculate the payoffs. *Shapley value* can be computed easily by using a formula regardless of whether or not the core is empty. However, when the core is non-empty, Shapley value may not be in the core and under some conditions the allocation scheme in terms of Shapley value may result in an unstable grand coalition.

An alternative solution concept known as the *nucleolus* was introduced by Schmeidler [17] in 1969 who proposed an allocation scheme that minimizes the "unhappiness" of the most unhappy player. Schmeidler [17] defines "unhappiness" (or, "excess") of a coalition as the difference between what the members of the coalition could get by themselves and what they are actually getting if they accept the allocations suggested by a solution. It was shown by Schmeidler [17] that if the core for a cooperative game is non-empty, then the nucleolus is always located inside the core and thus assures stability of the grand coalition. Unfortunately, unlike the Shapley value, there exists no closed-form formula for the nucleolus solution which has to be computed numerically in an iterative manner by solving a series of linear programming (LP) problems, or by solving a very large-scale LP problem (see, for example, Owen [14] and Wang [22]

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for textbook descriptions of these methods). The objective of this paper is to present analytic expressions to calculate the nucleolus solution directly without the need for iterative calculations that involve the solution of linear programs.

The nucleolus solution is an important concept in cooperative game theory even though it is not easy to calculate. As Maschler et al. [11, p. 336] pointed out, the nucleolus satisfies some desirable properties—e.g., it always exists uniquely in the core if the core is non-empty, and is therefore considered an important fair division scheme. As a consequence, some researchers have used this concept to analyze business and management problems; but, due to the complexity of the calculations, the nucleolus has not been extensively used to solve allocation-related problems. As an early application of the nucleolus concept, Barton [1] suggested the nucleolus solution as the mechanism to allocate joint costs among entities who share a common resource. Barton showed that using the nucleolus for this allocation problem can reduce the possibility that one or more entities may wish to withdraw from the resource-sharing arrangement. For other publications concerning the applications of the nucleolus, see, e.g., Du et al. [4], Gow and Thomas [7], and Leng and Parlar [10].

An n-player game in characteristic-function form (as originally formulated by von Neumann and Morgenstern [21, Ch. VI]) is defined by the set $N = \{1, 2, \ldots, n\}$ and a function $v(\cdot)$ which, for any subset (i.e., coalition) $S \subseteq N$ gives a number $v(S)$ called the value of S (see, also, Straffin [20, Ch. 23]). The characteristic value of the coalition S, denoted by $v(S)$, is the payoff that all players in the coalition S can jointly obtain. For a characteristic function game (N, v), let x_i represent an imputation (i.e., a payoff) for player $i = 1, 2, \ldots, n$. The nucleolus solution is defined as an n-tuple imputation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that the excess ("unhappiness") $e_S(\mathbf{x}) = v(S) - \sum_{i \in S} x_i$ of any possible coalition S cannot be lowered without increasing any other greater excess; see, Schmeidler [17]. With this definition, we find that the nucleolus of a cooperative game is a solution concept that makes the largest unhappiness of the coalitions as small as possible, or, equivalently, minimizes the worst inequity. In the sequential LP method that is based on lexicographic ordering (Maschler et al. [11]), to find the nucleolus solution we first reduce the largest excess $\max\{e_S(x)$, for all $S \subseteq N\}$ as much as possible, then decrease the second largest excess as much as possible, and continue this process until the n-tuple imputation $\mathbf{x}$ is determined.

Existing solution methods for the nucleolus either solve a series of linear programming (LP) problems or a single, but very large LP; see, Table 1. The description of the methods to find the nucleolus as summarized in Table 1 shows that most LP-based methods are iterative in nature and when they are not iterative, the resulting LP can be quite large (as in Kohlberg [9] and Owen [13]).

In this paper we focus on three-player cooperative games in characteristic-function form, and present an algebraic method that determines the nucleolus analytically (i.e., using closed-form expressions) without the need for iterative algorithms. We first present our analysis for the relatively simpler case of a cooperative game with empty core. If the core of a three-player cooperative game in characteristic function form is empty, then the nucleolus solution $\mathbf{y} = (y_1, y_2, y_3)$ is computed as,

$$y_i = \frac{v(123) + v(ij) + v(ik) - 2v(jk)}{3}, \quad \text{for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k.$$

We next derive the formulas that are used to compute the nucleolus solution for a three-player cooperative game with a non-empty core. For a three-player, nonempty-core cooperative game in characteristic function form, the nucleolus solution $\mathbf{y} = (y_1, y_2, y_3)$ can be computed as follows:

1. If $v(123) \geq 3v(ij)$, for $i, j = 1, 2, 3$ and $i \neq j$, then $y_1 = y_2 = y_3 = \frac{1}{3}v(123)$.
2. If $v(123) \geq v(ij) + 2v(ik)$, $v(123) \geq v(ij) + 2v(jk)$ and $v(123) \leq 3v(ij)$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, then $y_i = y_j = \frac{1}{4}[v(123) + v(ij)]$ and $y_k = \frac{1}{2}[v(123) - v(ij)]$.
3. If $v(123) \leq v(ij) + 2v(ik)$, $v(123) \geq v(ij) + 2v(jk)$ and $v(ij) \geq v(ik)$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, then $y_i = \frac{1}{2}[v(ij) + v(ik)]$, $y_j = \frac{1}{2}[v(123) - v(ik)]$, and $y_k = \frac{1}{2}[v(123) - v(ij)]$.

Year	Author(s)	Brief Description of Major Algorithms in the LP Method
1972	Kohlberg [9]	When the set of payoff vectors is a polytope, the nucleolus can be obtained as the solution of a single LP problem with n variables and $(2^n)!$ constraints.
1974	Owen [13]	When the set of payoff vectors is a polytope, the nucleolus can be obtained as the solution of a single LP problem with $2^{n+1}+n$ variables and 4^n+1 constraints.
1979	Maschler, Peleg and Shapley [11]	The nucleolus was characterized as the lexicographic center of a cooperative game, and it can be found by solving a series of $O(4^n)$ minimization LP problems with constraint coefficients of either -1, 0 or 1.
1981	Behringer [2]	Simplex based algorithm developed for general lexicographically extended linear maxmin problems to find the nucleolus by solving a sequence of $O(2^n)$ LP problems.
1981	Dragan [3]	Using the concept of coalition array, linear programs with only $O(n)$ rows and $O(2^n)$ columns are used to find the nucleolus solution.
1991	Sankaran [16]	Algorithm to find the nucleolus solution by solving a sequence of $O(2^n)$ LP problems. However, this method needs more constraints than in Behringer [2].
1994	Solymosi and Raghavan [19]	Algorithm to determine the nucleolus of an assignment game. In an (m,n)-person assignment game, the nucleolus is found in at most $m(m+3)/2$ steps, each one requiring at most $O(mn)$ elementary operations.
1996	Potters, Reijnierse and Ansing [15]	The nucleolus solution can be found by solving at most $n-1$ linear programs with at most 2^n-1 rows and 2^n+n-1 columns.
1997	Fromen [5]	By utilizing Behringer's algorithm [2], the number of LP problems to find the nucleolus is reduced to $O(n)$.

Table 1: A brief review of important algorithms to compute the nucleolus using the LP method.

4. If $v(123)+v(ij) \geq 2[v(ik)+v(jk)]$, $v(123) \leq v(ij)+2v(ik)$ and $v(123) \leq v(ij)+2v(jk)$, for $i,j,k = 1,2,3$ and $i \neq j \neq k$, then

$$y_i = \frac{1}{4}\{v(123)+v(ij)+2[v(ik)-v(jk)]\},\ y_j = \frac{1}{4}\{v(123)+v(ij)+2[v(jk)-v(ik)]\},$$
$$y_k = \frac{1}{2}[v(123)-v(ij)].$$

5. If $v(123)+v(ij) \leq 2[v(ik)+v(jk)]$, for $i,j,k = 1,2,3$ and $i \neq j \neq k$, then

$$y_i = \frac{1}{3}\{v(123)+v(ij)+v(ik)-2v(jk)\},\ y_j = \frac{1}{3}\{v(123)+v(ij)+v(jk)-2v(ik)\},$$
$$y_k = \frac{1}{3}\{v(123)+v(ik)+v(jk)-2v(ij)\}.$$

In this paper, we propose an algebraic method that gives the nucleolus analytically, to simplify the computations in calculating the nucleolus. This paper focuses on a three-player cooperative game. Only a single formula is needed for computing the nucleolus solution when the core of a three-player game is empty; and, for the nonempty-core game, we need some formulas each used under three specific conditions.

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