Portfolio Management with Stochastic Volatility

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Abstract

In this paper, we study option pricing in an incomplete market with stochastic volatility. We consider constant relative risk aversion (CRRA) preference. Under the power utility preference, we show that the fair price of the option depends on the ratio of the position in options holding to the wealth of the portfolio. Furthermore, we give an explicit expression for the market-price-of-risk. In addition, we show numerical examples to illustrate various points.

Key word: option pricing, stochastic volatility, power utility function.

1 Introduction

We study the price of option under constant relative risk aversion (CRRA) preferences for the case in which the underlying security has stochastic volatility. The most important phenomena are that the underlying dynamic equations are nonlinear and the option price depends on the position of the portfolio.

The path-breaking work of Black and Scholes (1973) on option price has wide applications in modern finance. This theory is for pricing options in a complete market. However, the empirical study on financial data showed the stochastic feature of volatility. We study

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option pricing when the underlying stock prices evolve with stochastic volatility in this paper. A model for stochastic volatility was first introduced by Hull and White (1987) and further major advances in stochastic volatility models can be found in Stein and Stein (1991) and Heston (1993). Stochastic volatility models can explain the patterns that are missing from the Black-Scholes' theory for pricing options in complete markets, such as "smile" and "term structure" of implied volatility. Stochastic volatility leads to an incomplete market which causes difficulties in theoretical study. In a financial market with stochastic volatility, the classical dynamic replication strategy which is critical in Black-Scholes theory is no longer possible. Arbitrage-free pricing approach does not give a unique answer for the option prices because many martingale probability measures exist. Selecting a particular measure is equivalent to specifying the market price of risk.

Expected utility maximization is one of the approaches to determine the market price of risk. Under this frame work, Hodges and Neuberger (1989) first introduced a notation of indifferent price. Namely, at this particular price, whether or not to purchase additional one unit of the option is indifferent in terms of expected utility. This approach has been adopted by many studies on stochastic volatility in the literatures (Musiela and Zariphopoulou (2001a,b), Stoikov and Zariphopoulou (2005)). A nice summary of main results on stochastic volatility can be found in the book of Lewis (2000). Davis (1999) suggested the notation of fair price, which is the utility indifferent price of holding an infinitesimal position in option.

Yang (2006) and Stoikov (2006) have further extended the concepts of fair price and indifferent price to a portfolio. They have carried out a detailed study of pricing option in a stochastic volatility setting under exponential utility function and demonstrated many interesting features. The exponential utility function has the property that the price of the option is independent of the wealth. This gives advantages that the partial differential equations are simpler. It has been shown that, based on the exponential utility function, there is maximum amount of wealth allocated in option market for his/her portfolio. No matter how big the portfolio is, the investor will never allocate more wealth than this maximum. This is the same phenomenon observed in the optimal strategy for the pure stock investment problem based on the expected exponential utility maximization: an investor's allocation of wealth in stock market will never be larger than a constant no matter how rich he/she is. Intuitively, one would expect that a wealthier investor will have more wealth invested in stock than an investor with less wealth. The CRRA utility function provides an optimal investment strategy which is consistent with this intuition. For the pure stock investment problem, the optimal strategy based on the CRRA utility function has the property that if the wealth of person A is λ times larger than that of person B, person A's optimal holding in stock will be λ times larger than that of person B as well. It has been shown that such property also hold in the stochastic volatility environment (see Liu, 2007). This motivates us to study option pricing with stochastic volatility based on CRRA utility function.

We study option price with stochastic volatility under CRRA utility function. We show that the fair price and the utility-indifference price of options depend on the current holding of risky assets in the portfolio and the current wealth. Our study showed that the option based power utility function still has the same important features demonstrated in the approach based exponential utility function (Yang, 2006). Namely, we can derive a set of nonlinear equations for the option price and give an explicit expression for the market-price-of-risk. We explain how the option price depends on the position of the portfolio.

This paper is organized as follows. In Section 2, we derive the dynamic equations which govern the price of options with stochastic volatility based on power utility function. Numerical examples are given in Section 3. Section 5 is for conclusion.

2 Theoretical formulation

In this section, we give a theoretical formulation for pricing European options on an underlying stock with stochastic volatility. Such a financial market is an incomplete market since one can not perfectly replicate the payoff of the option. For pricing options in incomplete markets, arbitrage argument will not uniquely determine the prices of options. An additional criteria is needed for pricing options. Here, we will follow the approach of maximization of expected power utility function. More specifically, we will derivate a set of nonlinear partial differential equations which determine the prices of European options.

We consider a financial market which consists of $N+2$ instruments: a riskless money market,

$$dP_t = rP_t\,dt, \tag{1}$$

where P_t is the value of the riskless instrument at time t and r is a constant interest rate; a risky stock of price S_t at time t modeled by

$$dS_t = \mu(v_t)S_t\,dt + \sqrt{v_t}S_t\,dB_t^S, \tag{2}$$

where μ is the drift, and $v^{\frac{1}{2}}$ is the stochastic volatility governed by the Heston's volatility process

$$dv = \kappa(\theta - v)dt + \xi v^{\frac{1}{2}}dB^v, \tag{3}$$

where θ, κ and ξ are constants, and θ is the mean of the variance v; and N European options $F^i(t, S_t, v_t, K_i)$, $i = 1, 2, \cdots, N$, written on the stock S, with a payoff $F^i(T_i, S_{T_i}, K_i)$. Here K_i is the strike price and T_i is the maturity date of the option F^i. In Eqs. (2) and (3), the two standard Brownian motions dB_t^S and dB_t^v are correlated ($dB_t^S\, dB_t^v = \rho\, dt$), and a, b, μ and ρ are functions of t, S_t and v_t.

We assume that the stock may pay a continuous dividend yield q, and that the trading allows unlimited lending and borrowing. Consider a portfolio of wealth W_t, which holds n_0 shares of stock and n_i units of option F^i at time t, then the change of wealth dW_t during the infinitesimal time interval $[t, t + dt]$ is

$$dW_t = n_0\, dS_t + qn_0 S_t\, dt + \sum_{i=1}^{N} n_i dF^i + r\left(W_t - n_0 S_t - \sum_{i=1}^{N} n_i F^i\right) dt. \tag{4}$$

For the sake of simplifying the mathematical expressions to be presented in this paper, we introduce discounted financial instruments and discounted wealth:

$$\begin{aligned} s = e^{-rt} S_t, \quad p = e^{-rt} P_t, \quad k_i = e^{-rT_i} K_i, \\ w = e^{-rt} W_t, \quad f^i(t, s, v) = e^{-rt} F^i(t, S_t, v). \end{aligned} \tag{5}$$

Then from Eqs. (2), the stochastic equation for the discounted stock is

$$ds = \nu(v) s\, dt + v^{\frac{1}{2}} s\, dB^s, \tag{6}$$

where $\nu = \mu - r$ is the discounted drift. The discounted budget equation becomes

$$dw = n_0\, ds + qn_0 s\, dt + \sum_{i=1}^{N} n_i\, df^i. \tag{7}$$

Our goal is, for given t, s and v, to determine the price of the discounted options f^i, $i = 1, 2, \cdots, N$. We will follow the approach of maximization of expected wealth under the power utility given by

$$U[w(T)] = \frac{1}{\gamma}[w(T)]^{\gamma} \tag{8}$$

to determine option prices. Here the dimensionless parameter γ models an investor attitude towards risk. $\gamma < 1$ is for a risk aversion investor and $\gamma = 1$ is for a risk neutral investor. The smaller γ is, the more risk averse the investor is. One can replace w^{γ} by $w^{\gamma} - 1$ in the definition of power utility function, since all utility functions related by an affine transformation are equivalent. Then the limit $\gamma \to 0$ corresponds to the log utility function, $U(w) = log(w)$. The relative risk aversion function of the power utility function is $1 - \gamma$, thus the power utility function is also called the constant relative risk aversion (CRRA) utility function. We assume that the wealth never becomes negative. This constraint is satisfied automatically for power utility functions and needs not be imposed separately.

Since we expect that, based on the power utility function, a person's optimal holding in i-th option will be proportional to his/her wealth, it will be more convenient to choose $m_0 = \frac{n_0}{w}$ and $m_i = \frac{n_i}{w}$ $(i = 1, 2, \cdots, N)$, instead of n_0 and n_i, as control variables. In terms of m_0 and m_i, the discounted budget equation Eq. (7) becomes

$$\frac{dw}{w} = m_0\, ds + q m_0 s\, dt + \sum_{i=1}^{N} m_i\, df^i \tag{9}$$

Applying Itô's lemma to $f^i(t, s, v)$ gives

$$df^i = \left(f^i_t + \nu s f^i_s + \kappa(\theta - v) f^i_v + \mathcal{O}_2 f^i\right)\, dt + v^{\frac{1}{2}} s f^i_s\, dB^s_t + \xi v^{\frac{1}{2}} f^i_v\, dB^v_t \tag{10}$$

where the second-order differential operator $\mathcal{O}_2$ is defined as

$$\mathcal{O}_2 \psi(t, s, v) \doteq \frac{1}{2} v s^2 \psi_{ss} + \frac{1}{2} \xi^2 v \psi_{vv} + \rho \xi v s \psi_{sv}$$

The valuation function J, which is the maximized expected utility conditioned on the current state information, is defined as

$$J(t, w, s, v) \doteq \sup_{m_0, m_i} E\left[U\left(w_T\right)\right]. \tag{11}$$

Obviously, the investment horizon T should be larger than the longest maturity date T_i. An application of Hamilton-Jacobi-Bellman (HJB) equation to Eq. (11) gives

$$\sup_{m_0, m_i} \mathcal{L} J = 0. \tag{12}$$

The conditions for optimality are

$$\frac{\partial}{\partial m_0} \mathcal{L} J = 0, \tag{13}$$

and

$$\frac{\partial}{\partial m_i} \mathcal{L} J = 0. \tag{14}$$

In Eqs. (12)-(14), the expression for $\mathcal{L} J$ is extremely complicated and long. Following Yang (2006), to make the expressions compact, we define a shorthand notation $C_{(*,*)}$ for the coefficients in a stochastic differential equation,

$$dZ_t = C_{(Z,t)}\, dt + C_{(Z,x)}\, dB^x_t + C_{(Z,y)}\, dB^y_t + ...$$

Thus, we have

$$\begin{aligned}\mathcal{L}J \doteq\ & J_t + C_{(s,t)}J_s + C_{(v,t)}J_v + C_{(w,t)}J_w \\ & + \frac{1}{2}C^2_{(s,s)}J_{ss} + \frac{1}{2}C^2_{(v,v)}J_{vv} + \rho C_{(s,s)}C_{(v,v)}J_{sv} \\ & + \frac{1}{2}\left[C^2_{(w,s)} + 2\rho C_{(w,s)}C_{(w,v)} + C^2_{(w,v)}\right]J_{ww} \\ & + \left[C_{(w,s)} + \rho C_{(w,v)}\right]C_{(s,s)}J_{ws} + \left[\rho C_{(w,s)} + C_{(w,v)}\right]C_{(v,v)}J_{wv}\end{aligned} \tag{15}$$

where the subscripts on J denote partial derivatives.

J is conjectured to have the form:

$$J = \frac{1}{\gamma}w^{\gamma}\phi(t,s,v) \tag{16}$$

Based on this functional form, after performing tedious manipulations, Eqs. (13), (14) and (12) become the following three nonlinear partial differential equations respectively,

$$\pi = m_0 s = \frac{1}{1-\gamma}\left(\chi + s\frac{\phi_s}{\phi} + \rho\xi\frac{\phi_v}{\phi}\right) - \left(s\sum_{i=1}^{N} m_i f^i_s + \rho\xi\sum_{i=1}^{N} m_i f^i_v\right), \tag{17}$$

$$f^i_t - qsf^i_s + \mathcal{O}_2 f^i + \left[\kappa(\theta - v) - \rho\chi\xi v + \left(1-\rho^2\right)\xi^2 v\left(\frac{\phi_v}{\phi} - (1-\gamma)\sum_{i=1}^{N} m_i f^i_v\right)\right] f^i_v = 0, \tag{18}$$

$$\begin{aligned}\frac{1}{\phi}\left[\phi_t + (\chi v - q)s\phi_s + \kappa(\theta - v)\phi_v + \mathcal{O}_2\phi\right] + \frac{\gamma v}{2(1-\gamma)}\left(\chi + s\frac{\phi_s}{\phi} + \rho\xi\frac{\phi_v}{\phi}\right)^2 & \\ + \frac{1}{2}\gamma(1-\gamma)\left(1-\rho^2\right)\xi^2 v\left(\sum_{i=1}^{N} m_i f^i_v\right)^2 &= 0,\end{aligned} \tag{19}$$

for $i = 1, 2, \cdots, N$, where $\chi = (\nu(v) + q)/v$ is assumed as constant. The final conditions for f^i and ϕ are

$$f^i(T_i, s, v) = g^i(T_i, s_{T_i}) \qquad \text{and} \qquad \phi(T, s, v) = 1 \tag{20}$$

respectively, where the investment horizon $T \geq \max T_i$ and $g^i(s_{T_i})$ is the discounted payoff of the option. We comment that the discounted current wealth w does not appear in Eqs. (17)-(20). The prices of options in incomplete markets based maximization of power utility function are determined by a set of coupled nonlinear PDEs (18) and (19) with the final conditions given by Eq. (20).

Extensive studies show that the option price under incomplete market models with stochastic volatility satisfies the following PDE

$$f^i_t - qsf^i_s + [\kappa(\theta - v) - \lambda\xi v^{\frac{1}{2}}]f^i_v + \mathcal{O}_2 f^i = 0 \tag{21}$$

θ	γ	ξ	κ	χ	ρ	τ	$\bar{v}$	v
0.2500	-1.0000	0.5000	1.0000	1.0000	-0.5000	1.0000	0.2500	0.2500

Table 1: The base point in the parameter space. $\rho = -0.5$ is for correlated case and $\rho = 0$ is for uncorrelated case. $\tau = T - t$ is the time to the maturity date. $\bar{v}$ and v denote, respectively, the mean of the stationary distribution and the initial value of variance.

where λ called market-price-of-risk is exogenously given. Our basic equation (18) for option can also be written in the form of Eq. (21) with λ endogenously given by

$$\lambda(t, s, v) = \rho\chi v^{\frac{1}{2}} - \left(1 - \rho^2\right) \xi v^{\frac{1}{2}} \left(\frac{\phi_v}{\phi} - (1 - \gamma) \sum_{i=1}^{N} m_i f_v^i \right). \tag{22}$$

Yang (2006) showed that, based on the exponential utility function, ϕ and consequently λ, depends on the position of the portfolio. In next section, we will show that the same feature occurs in our approach.

3 Numerical study

In this section, we present results of numerical study of option prices with stochastic volatility. The numerical results are determined by the solving nonlinear partial differential Eqs. (18)-(19) with the final conditions given by Eqs. (20). We will only consider European vanilla calls with discounted strikes k and time to maturities τ. We choose initial wealth w_0 as the unit for wealth. Then the control variables $m_0 w_0 = \frac{n_0}{w} w_0$ and $m_1 w_0 = \frac{n_1}{w} w_0$ are dimensionless. The parameters used in numerical computation are shown in Table 1. For the correlated case we choose $\rho = -0.5$ and for the uncorrelated case we choose $\rho = 0$. For the sake of simplicity is our illustration, s is set to 1. In addition, the dividend rate q is set to zero, because the effect of the dividend rate q can be eliminated by simple change of variables. The value of v is set to 0.25.

We investigate the impact of the at-the-money call option, which means that there is only one type of options with the strike price $k = s$. In Fig. 1, the dimensionless price of call option f/s is plotted against the position $\boldsymbol{m}$ for correlated and uncorrelated cases. From Fig. 1, it is clear that the option price is a decreasing function of the option's position $\boldsymbol{m}$ in the portfolio. Figure. 1 shows that adding more position in option to the portfolio lowers the fair price of the option; conversely reducing position in option raises the fair price. This is consistent with the practice in option trading.

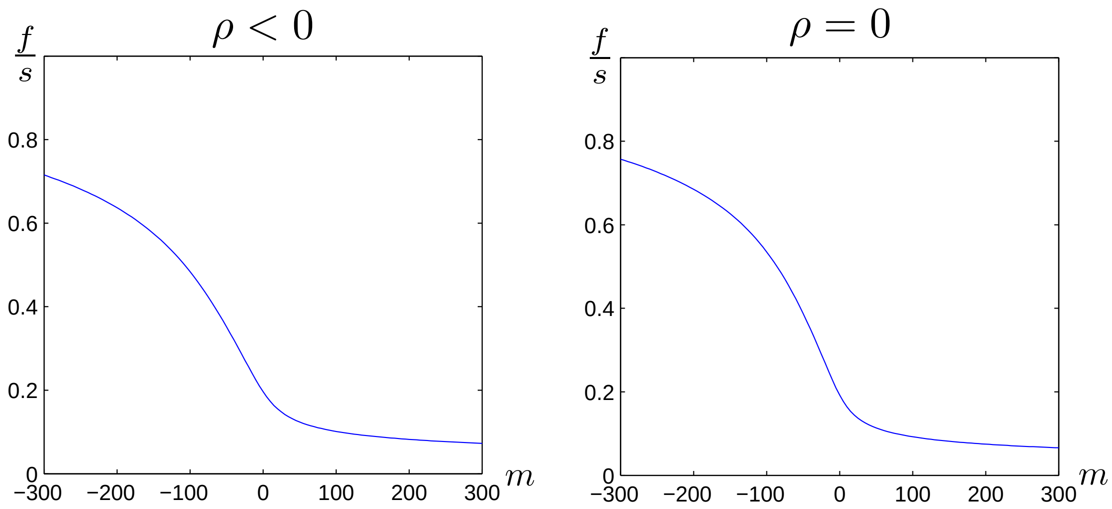

Figure 1: The option prices are plotted against the position m for both correlated and uncorrelated cases. It shows that the option price f depend on the position m.

4 Conclusions

In complete market, the option prices are determined by Black-Scholes formulae, which are independent of the position of a portfolio in options. It has been shown that under the exponential utility function (Yang, 2006), the option prices depend on the position of the options. In this paper, we show that this feature is also true for the power utility function.

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