Modeling portfolio rebalancing by modifying utility theory

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Abstract

Investors are divided into three groups in modern portfolio theory (MPT) according to their risk pr eferences, which are assumed to be invariant in any situations. However, as shown in many psycholo gical tests, most investors change their attitude toward risk when they are in different situations. Espec ially, most investors are risk seeing when getting a chance to recapture their losses in investments. MP T has not taken these features into consideration in its models for portfolio choices and portfolio rebal ancing.

This paper is to propose a model for rational behavior in portfolio rebalancing. We first redefine the notion of rational behavior in investment, and then define the utility of an investment as a function of profit to describe investors' behavior under loss. Since most investors are risk averse when invest ment is producing a profit and risk seeking when investment is causing a loss, we propose a conditio nal function as the utility function for rational investors, and build an optimization model for portfoli o rebalancing problems. Since our model captures investors' rational behavior more faithfully, rebalanc ing decision derived from this model should be more acceptable for investors.

Keywords: Portfolio rebalancing, utility theory, risk preference.

1. Introduction

The expected utility theory proposed by John von Neumann and Oskar Morgenstern(1944) provides a the oretical framework for decision making under risk, which is also employed in modern portfolio theory to analyze investors' decision under risk, wherein a rational investors is defined as a maximizer of the exp ected utility.

In choosing a utility function for an investor, his/her risk preference should be determined first. If an in vestor is willing to take higher risks to achieve average return, she/he is called to have risk seeking pref erence, while she/he is called to have risk aversion preference if she/he is not willing to take risks to a chieve average return, and risk neutral preference if she/he does not care risks when two risk choices wi ll achieve the same return. Investors with different risk preferences are assigned with different utility fun ctions, which are assumed to be invariant in any situations.

However, investors' risk preferences are not constant in many psychological tests, most investors show di fferent risk attitudes when they are in different situations. Most investors with risk aversion preference sh ow risk seeking preference when getting a chance to recapture their losses in investments. It is not proper to assume that a rational investor has an invariant risk preference.

This paper is to propose a model for rational behavior in portfolio rebalancing by taking these features into consideration. We first redefine the concept of rational behavior in investment, and then define the utility of investment as a function of profit to describe investors' behavior in different situations. Since most investors are risk averse when investment is producing a profit and risk seeking when investment is causing a loss, we propose a conditional function as the utility function of rational investors, and build a model for portfolio rebalancing problems based on this utility function.

2. Risk preference and rational behavior in investment

It is well known that investors' risk preference plays an important role in investment decision, and an investor's risk preference is determined by his/her choice in the following situation:

Lottery A: Only an outcome with a value of V
Lottery B: Two outcomes are possible whose average value is V

If an investor prefers Lottery A to Lottery B, then he/she is said to have risk aversion preference, if he/she prefers Lottery B to Lottery A, he/she has risk seeking preference, and if the Lottery A and Lottery B are indifferent to the investor, he/she is said to have risk neutral preference.

For each type of risk preference, a utility function is used to describe an investor's feature quantitatively. A risk aversion investor is described by a concave function, while a risk seeking investor is assigned with a convex function, and a risk neutral investor is described by a linear function, as shown in Figure 1 below.

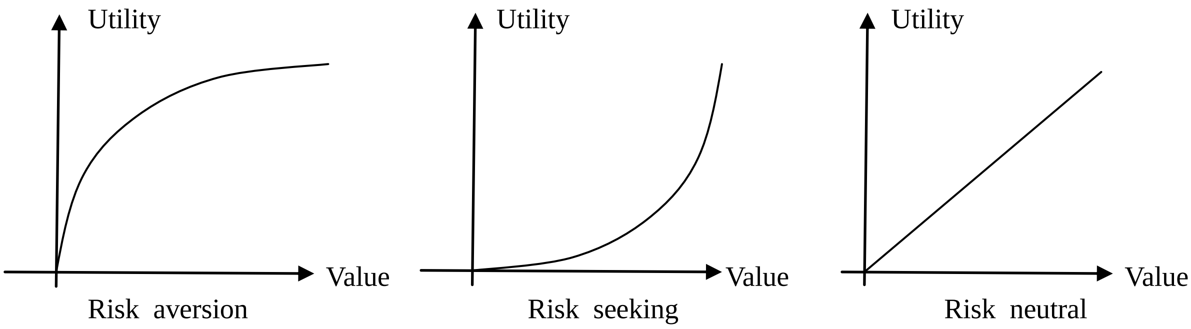

Figure 1: Utility functions for investors with different risk preference

Denote the decision variable by x, the values of risk factors by θ. The outcome of decision is a function of x and θ, denote it by $g(x, \theta)$, while the utility of the outcome is defined as a function of $g(x, \theta)$, let it be $u(g(x, \theta))$. Then rational investor is assumed to make decision by maximizing the expected utility of $g(x, \theta)$, that is, the solution of the following optimization model is taken as a rational decision:

$$\mathrm{Max}_{x \in X} E_{\theta}\, u(g(x, \theta)) \qquad (2.1)$$

where X is a set of feasible portfolios. In the above framework, the investor's risk preference is usually taken as invariant.

However, most investors with risk aversion preference show risk seeking preference when getting a chance to recapture their losses in investments, thus it is not proper to assume that a rational investor has an invariant risk preference. This feature should be taken into account in describing investors' rational behavior, especially in dealing with portfolio rebalancing problems wherein investors may in the red.

We assume that a rational investor is two sides, his/her risk preference is risk aversion when he/she is i n the black, and risk seeking when he/she is in the red.

In order to reflect this modification in utility function, we define utility as a function of profit or profit rate, and use a conditional function as the utility function of rational investors, as shown in Figure 2 bel ow.

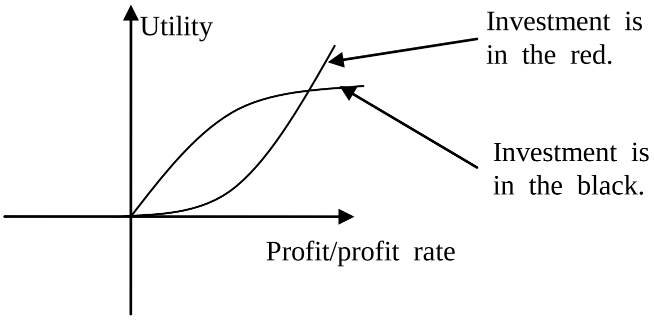

Figure 2: Utility function for rational investors

The utility function in Figure 2 is a conditional function, it is concave when investors is in the black a nd convex when he/she is in the red.

We use a symbolic variable K to express the status of investment with K=0 and K=1 showing that curr ent investment is in the black and in the red, respectively. Then the utility function of rational investo rs can be written as follows,

$$u(y) = \begin{cases} u_1(y), & \text{if } K = 0 \\ u_2(y), & \text{if } K = 1 \end{cases} \tag{2.2}$$

where y stands for the profit or profit rate from investment, u_1 is a concave function and u_2 is a conve x function. Various functions are possible choices for u_1 and u_2, a simple form is given in the followin g formula,

$$u(y) = \begin{cases} \lambda_1 y^{\alpha}, & \text{if } K = 0 \\ \lambda_2 y^{\beta}, & \text{if } K = 1 \end{cases}, 0 < \alpha < 1, \beta > 1, \lambda_1, \lambda_2 > 0 \tag{2.3}$$

where $\alpha, \beta, \lambda_1, \lambda_2$ are parameters representing the risk preference of investors.

3. Model for portfolio rebalancing problems

Consider the following portfolio rebalancing problem:

An investor wants to adjust his portfolio because his anticipation on future market is updated.

Denote the initial portfolio by $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$, and current portfolio by $x^1 = (x_1^1, x_2^1, \cdots, x_n^1)$, where x_i^0 and x_i^1 stand for the investment ratio on asset-i in the two portfolios respectively.

Let the value of the initial portfolio and current portfolio be W^0 and W^1 respectively. Then the utility function of a rational investor is assumed as follows:

$$u(y) = \begin{cases} u_1(y), & \text{if } W^1 \geq W^0 \\ u_2(y), & \text{if } W^1 < W^0 \end{cases}$$

Let R_i be the profit rate of asset-i in future market, which is usually taken as a stochastic variable, then the profit rate of portfolio $x = (x_1, x_2, \cdots, x_n)$ is

$$g(x, \theta) = \sum_{i=1}^{n} x_i R_i \tag{3.1}$$

and the expected utility of portfolio x is

$$E_\theta\, u\big(g(x, \theta)\big) = E_R u(\sum_{i=1}^{n} x_i R_i)$$

When an investor is in the red, the rebalanced portfolio is the solution of the following optimization model:

$$\mathrm{Max}_{x \in X_r}\ E_R\ u_2\left(\sum_{i=1}^{n} x_i R_i\right) \tag{3.2}$$

where X_r be a set of feasible portfolios which is determined by considering other constraints such as rebalancing cost constraint.

When an investor is in the black, the rebalanced portfolio is the solution of the following optimization model:

$$\mathrm{Max}_{x \in X_r}\ E_R\ u_1\left(\sum_{i=1}^{n} x_i R_i\right) \tag{3.3}$$

To solve these models, one direct way is to identify the utility function of an investor first and then solve these models using optimization techniques. Identifying utility function is a classic issue in economics and decision theory, a large number of literature can be found by searching the web.

When the utility function has certain special features, or the distribution of the stochastic variables is of a special form thus as the normal distribution, we may solve these models indirectly following the similar reasoning introduced in Tabata(1993).

We will report our results on solving these rebalancing models elsewhere.

4. Conclusions and Remarks

Investors' risk preferences are not invariant in making investment decisions, most risk aversion investors are risk seeing when getting a chance to recapture their losses in investments, this paper modified the notion of rational investors by taking this feature into consideration, and used a conditional function as the utility function of investors.

We also modeled portfolio rebalancing problems with the conditional utility function to reflect investors' desire more faithfully, rebalancing decision derived from this model is expected to be more acceptable for investors.

Portfolio rebalancing problems are usually handled by explicitly considering the risk and return from investment as in Xu, etc(2008), the merit of this way is that investors' requirements on risk and return can be considered directly. However, there is no consensus on the indicator for risk measure, see Dowd(2002) for an introduction of various risk measurement, different modeling methods result in different rebalancing decisions, which usually makes investors puzzled. Modeling portfolio rebalancing problems with the modified utility function is another alternative, this method does not require investors to determine their requirements on risk and return, but we need to identify the utility function or we need to make some strong assumptions on the utility function.

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